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Research article

Hyers-Ulam-Mittag-Leffler stability of fractional differential equations with two caputo derivative using fractional fourier transform

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Abstract: In this paper, we discuss standard approaches to the Hyers-Ulam Mittag Leffler problem of fractional derivatives and nonlinear fractional integrals (simply called nonlinear fractional differential equation), namely two Caputo fractional derivatives using a fractional Fourier transform. We prove the basic properties of derivatives including the rules for their properties and the conditions for the equivalence of various definitions. Further, we give a brief basic Hyers-Ulam Mittag Leffler problem method for the solving of linear fractional differential equations using fractional Fourier transform and mention the limits of their usability. In particular, we formulate the theorem describing the structure of the Hyers-Ulam Mittag Leffler problem for linear two-term equations. In particular, we derive the two Caputo fractional derivative step response functions of those generalized systems. Finally, we consider some physical examples, in the particular fractional differential equation and the fractional Fourier transform.

Keywords: fractional fourier transform; fractional differential equation; Hyers-Ulam-Mittag-Leffler stability; Mittag-Leffler function; Caputo derivative

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1. Introduction

In recent years, the area related to fractional differential and integral equations has received much attention from numerous mathematicians and specialists. The derivatives of fractional order represent physical models of multiple phenomena in different fields such as biology, physics, mechanics, dynamical systems, and so on (see [1–6] and the references therein).

The possibility of fractional calculus was presented in 1695, when the notation $\frac{d^{\tau}}{dt^{\tau}}h(r)$ was introduced to indicate the derivative of function $h(r)$ in order τ. Specifically, Leibniz composed a letter to LHospital in which he posed an enquiry on the derivative of order $\tau = 1, 2$ which led to the establishment of fractional calculus. Later on, the fractional derivative was presented by Lacroix [7]. Perhaps the most utilized fractional derivatives are Riemann Liouville (RL) and Caputo derivatives, which assume an immodest role in fractional order differential equation.

One of the best examination regions in fractional order differential equation, which receives vast considerations by analysts, entails the existence theory of solution. For details concerning the present hypothesis, see [8–12]. Finding an exact solution of fractional order differential equation is exceptionally difficult and the type of exact solution is regularly is important to study an approximate solution with a relatively simple form and examine how close both the approximate and exact solutions are. Overall, we state that a fractional-order differential equation is said to be Hyers-Ulam (HU) stable if, for every solution of the fractional-order differential equation, there exists an approximate solution of the concerned equation that is close to it.

Ulam [13] formulated the stability of a functional equation, which was solved by Hyers [14] using an additive function defined on the Banach space. This result led Rassias [15] to study and generalize the stability concept, establishing the Hyers-Ulam-Rassias stability. An integral transform (introduced by Fourier) involves a trigonometric form of the Mittag-Leffler function to identify an analytic solution concerning a differential equation of fractional order. The Fourier transforms, Mittag-Leffler function, and fractional trigonometric function constitute an effective tool for analytic expression of the solution of differential equation of non-integer order. Indeed, the Fourier transform has become popular because of recent developments in differential applications. It is also seen as the easiest and most effective way among many other transforms. Luchko [16] defined the fractional Fourier transform (FRFT) of real order $\tau, 0 < \tau \leq 1$ and discussed its important properties. The application of fractional Fourier transform for undertaking certain types of differential equations of fractional order has also been conducted. Indeed, there are many studies on fractional Fourier transform and its applications in the literature [17–19].

In 2017, Wang et al. [20] discussed the stability of fractional differential equation based on the right-sided RiemannLiouville fractional derivatives for continuous function space. The fixed point theorem and weighted space method were exploited. In [21], a study on the HU stability condition was conducted, focusing on an impulsive R-L fractional neutral functional stochastic differential equation with time delays. In [22], the stability criteria of a class of fractional differential equations were investigated, in which the Krasnoselskii fixed point method was employed. Recently, Upadhyay et al. [23] discussed the RL fractional differential equations using the Hankel transform method. At present, some remarkable results to the stability of fractional differential equations have been reported (see [24–26] and the references therein). In [27, 28], the author studied the Hyers-Ulam stability of linear differential equation by using Fourier transform. To the best of our knowledge, there are no results on Hyers-Ulam stability of fractional differential equation by fractional Fourier transform. Some important works related to the recent development in fractional calculus and its applications should be discussed in [29–31].

Motivated by the ongoing research in this field, we examine the Hyers-Ulam stability and generalized Hyers-Ulam stability of fractional order differential equation in this study becomes

$$\left(\mathcal{D}_{\vartheta}^{\tau}h\right)(r) = G(r), \ \forall\ r \in R,$$

and the delay differential equation of fractional order

$$\left(\mathcal{D}_{\vartheta}^{\tau}h\right)(r-\xi) = G(r), \ \forall\ r \in R,$$

where $\mathcal{D}_{\vartheta}^{\tau}$ represents RL fractional derivative, $\xi > 0, \vartheta \in R$ and $0 < \tau \leq 1$ with the help of fractional Fourier transform.

In our investigation, we establish the fractional Fourier transform and present it in an integral form. Furthermore, using the convolution concept and properties of fractional Fourier transform, the solution of the stability conditions concerning fractional order differential equation is established. Specifically, we analyze Hyers-Ulam-Rassias stability of the nonlinear fractional order differential equation of the form

$$\left(\mathcal{D}_{\vartheta}^{\tau}h\right)(r) = G(r, h(r)), \ \forall\ r \in R,$$

and use the fixed point theorems for examining the existence and uniqueness of the solution.

The conduct of the analytical solutions of the fractional differential equation represented by the fractional-order derivative operators is the fundamental profession in numerous stability issues. Motivated by the usage of the Mittag-Leffler functions in many spaces of science and designing we present this paper.

The main aim of this paper is to prove the Hyer-Ulam-Mittag-Leffler stability of the following fractional differential equations using the fractional Fourier transform

$$\left({}^{c}\mathcal{D}_{0+}^{\tau}y\right)(\mathfrak{c}) - \lambda\left({}^{c}\mathcal{D}_{0+}^{\delta}y\right)(\mathfrak{c}) = h(\mathfrak{c}), \tag{1.1}$$

and

$$\left({}^{c}\mathcal{D}_{0+}^{\tau}y\right)(\mathfrak{c}) - \lambda\left({}^{c}\mathcal{D}_{0+}^{\delta}y\right)y(\mathfrak{c}) - h(\mathfrak{c}) = F(\mathfrak{c}), \tag{1.2}$$

where $s > 0, \lambda \in \mathcal{R}, p-1 < \tau \leq p, q-1 < \delta \leq q, 0 < \delta < \tau, p, q \in \mathcal{N} \leq p$, $h(\mathfrak{c})$ and $h(\mathfrak{c})$ real functions defined on $\mathcal{R}_{+}$, and ${}^{c}\mathcal{D}_{0+}^{\tau}$ is the Caputo fractional derivative of order τ defined by

$$\left({}^{c}\mathcal{D}_{0+}^{\tau}\right) = \frac{1}{\Gamma(p-\tau)}\int_{0}^{\mathfrak{c}}(\mathfrak{c}-r)^{p-\tau-1}y^{(p)}(r)dr. \tag{1.3}$$

We organize this article as follows. The related fundamental properties, lemmas and definitions are presented in section 2. In section 3, Hyers-Ulam-Mittag-Leffler stability of fractional-order linear differential equation and non-linear differential equation is explained. Numerical examples and conclusions are given in section 4.

2. Preliminaries

The Fourier transform is used, for solving partial differential equations. We will need it only in some applications of fractional calculus so we only give the most important formulas. For further facts, we recommend the same books as for the Laplace transform [2].

Let $f(x)$ be a real function of one real variable, such that its Lebesgue integral over the real numbers converges and such that $f(x)$ with its derivative are piecewise continuous. Then the Fourier image of the function $f(x)$, we denote it by $\hat{f}(k) = F\{f(x), x, k\}$.

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx.$$

For Fourier images we will use same letters like for the original function with hat and the variable k.

Consider $L^1(R)$ as the space related to the complex-valued Lebesgue integrable function on the real line R with norm

$$||h|| = \int_R |h(r)|dr.$$

The definition of a Fourier transform with respect to a function $h \in L^1(R)$ is

$$\widehat{\mathcal{H}}(\omega) = (Fh(r))(\omega) = \int_{\infty}^{-\infty} h(r)e^{i\omega r}dr, \ \forall\ \omega \in R.$$

The form of the associated inverse Fourier transform is

$$h(r) = (F^{-1}\widehat{\mathcal{H}}(\omega))(r) = \frac{1}{2\pi}\int_{\infty}^{-\infty} \widehat{\mathcal{H}}(\omega)e^{-i\omega r}d\omega, \ \forall\ r \in \mathcal{R}.$$

Note that Fourier transform is useful for conversion of a function between the time and frequency domains. It adopts the principle of rotation operation on the time-frequency distribution.

Definition 1. *Given parameter τ, we can express the fractional Fourier transform of function $h(r)$ in a one-dimensional case as follows [32]:*

$$\widehat{\mathcal{H}}(\omega) = (F_\alpha h(r))(\omega) = \int_{\infty}^{-\infty} h(r)K\tau(r,\omega)dr,$$

where kernel $K\tau(r,\omega)$ is

$$K\tau(r,\omega) = \begin{cases} B\tau e^{\frac{i(r^2+\omega^2)cot\tau}{2} - ir\omega cosec\tau}, & \tau \neq n\pi, \\ \frac{1}{2\pi}e^{-ir\omega}, & \tau = \frac{\pi}{2}, \end{cases}$$

and n is an integer, while

$$B_\tau = (2\pi i sin\tau)^{\frac{-1}{2}}e^{i\tau^2} = \frac{\sqrt{1-icot\tau}}{2\pi}.$$

As such, the form of the associated inverse fractional Fourier transform is

$$h(r) = \frac{1}{2\pi}\int_{\infty}^{-\infty} K\tau(r,\tau)\widehat{H}(\omega)d\omega,$$

where

$$K\tau(r,\tau) = \frac{(2\pi isin\tau)^{\frac{1}{2}}}{sin\tau} e^{\frac{-i\tau}{2}} e^{\frac{-i(r^2+\omega^2)cot\tau}{2}} + ir\omega cosec\tau = B_\tau^{'} e^{\frac{-i\tau}{2}} e^{\frac{-i(r^2+\omega^2)cot\tau}{2}} + ir\omega cosec\tau$$

$$B_\tau^{'} = \frac{(2\pi isin\tau)^{\frac{1}{2}}}{sin\tau} e^{\frac{-i\tau}{2}} = \sqrt{2\pi(1+icot\tau)}.$$

Definition 2. *The fractional trigonometric function is denoted by*

$$\mathbb{E}_\tau(ix^\tau) = cosr^\tau + isinr^\tau,$$

with

$$cosr^\tau = \sum_{k=0} \infty(-1)^k \frac{r^{2\tau}k}{\Gamma(1+2\tau k)} \text{ and } sinr^\tau = \sum_{k=0} \infty(-1)k \frac{x(2k+1)\tau}{\Gamma(1+\tau(2k+1))}.$$

Luchko et al. [33] introduced a new fractional Fourier transform F_τ of order $\tau, (0 < \tau \leq 1)$ and its definition is

$$\widehat{\mathcal{H}}\tau(\omega) = (F_\tau h)(\omega) = \int_\infty^{-\infty} h(r) e_\tau(\omega, r) dr,$$

where

$$e_\tau(\omega, r) = \begin{cases} \mathbb{E}_\tau(-i|\omega|^{1/\tau} r), & \omega \leq 0, \\ \mathbb{E}_\tau(i|\omega|^{1/\tau} r), & \omega \geq 0. \end{cases}$$

$$sign(\omega) = \begin{cases} -1, & \omega < 0, \\ 1, & \omega \geq 0. \end{cases}$$

As such, the definition of the associated inverse fractional Fourier transform is

$$h(r) = \frac{1}{2\pi\tau} \int_\infty^{-\infty} \mathbb{E}_\tau(-isign(\omega)|\omega|^{1/\tau} x)|\omega|^{\frac{1}{\tau}-1} \widehat{H}_\tau(\omega) d\omega,$$

for any $r \in R$ and $\tau > 0$. If $\tau = 1$, then $\widehat{\mathcal{H}}\tau(\omega)$ and the classical Fourier transform are the same.

Suppose that the space of a function with fast decrease is denoted as S. In other words, the following relation with respect to the space of infinity differentiable functions $v(r)$ on R is satisfied: Given $r \in R$ and $n, k \in N \cup \{0\}$. If $v(r) \in S \subseteq R$, then

$$\|v^k(r)\| \leq \frac{M}{|r|^n} \ \ (n, k \in N \cup \{0\},\ n > k; |r| \to \infty).$$

Based on $V(R)$, the following relation with respect to a set of functions $v \in S$ is satisfied:

$$\left.\frac{d^n v}{dr^n}\right|_{r=0} = 0, n = 0, 1, 2, 3, ...$$

The Lizorkin space is $\phi(R) \subset L^1(R)$ and it is defined as the Fourier pre-image of the space $V(R)$ in the space S of the form

$$\phi(R) = \{h \in S;\ F(h) \in V(R)\}.$$

The reason for using the Lizorkin space is its convenience in using the Fourier transform as well as the inverse Fourier transforms with fractional integration and differentiation operators. The properties and associated details of the Lizorkin space have been discussed in many studies (see [34–36]). In our study, we use F to represent the domain of either real R or complex C. According to the definition of Lizorkin space, the orthogonality condition is satisfied by any function $h \in (R)$. That is

$$\int_{\infty}^{\infty} r^n h(r)dr = 0,\ n = 0, 1, 2, 3, ...$$

Note that the property of the Fourier transform and its inverse holds for the space $\phi(R)$. In other words, both transforms are inverse of one another, that is,

$$F^{-1}Fh = h,\ h \in \phi(R).$$

Definition 3. *The function* $(h_1 * h_2)(r) = \int_R h_1(r\tau)h_2(\tau)d\tau$ *is denoted as the convolution of both functions of* h_1 *and* h_2 *defined on* $\phi(R)$.

Some properties of fractional Fourier transform that are closely related to the solution in this study are given as follows. Let h, h_1 *and* h_2 *be functions belonging to* $\phi(R)$. *Then*

(1) If $(F_\tau h_1)(\omega) = (F_\tau h_2)(\omega)$, *then* $h_1(r) = h_2(r)$,
(2) $F(F_\tau h(x - \xi))(\omega) = e_\tau(\omega, \xi)\tilde{H}(\omega)$,
(3) $F_\tau(h_1 * h_2)(\omega) = F_\tau((h_1)(\omega))F_\tau((h_2)(\omega)$,
(4) $F_\tau^{-1}(h_1h_2)(r) = F_\tau^{-1}(h_1)(r)) * F_\tau^{-1}(h_2)(r)$.

Definition 4. *[37] The definition of RiemannLiouville fractional integral of order* $\tau > 0$ *is*

$$\left(I_+^\tau h\right)(r) = \frac{1}{\Gamma(\tau)}\int_{-\infty}^{r}(rt)^{\tau-1}h(t)dt\ (Right\ RiemannLiouville\ fractional\ integral),$$

and

$$\left(I_-^\tau h\right)(r) = \frac{1}{\Gamma(\tau)}\int_{r}^{\infty}(tr)^{\tau-1}h(t)dt\ (Left\ RiemannLiouville\ fractional\ integral),$$

where $Re(\tau) > 0$, *we have* $\Gamma(\tau) = \int_0^\infty e^{-u}u^{\tau-1}du$.

Definition 5. *[37] The definition of RiemannLiouville fractional derivative of order* $\tau > 0$ *is*

$$(D_+^\tau h)(r) = \frac{d}{dr}(I_+^{1-\tau}h)(r)\ \ (Right\ RiemannLiouville\ fractional\ derivative),$$
$$(D^\tau h)(r) = -\frac{d}{dr}(I^{1-\tau}h)(r)\ \ (Left\ RiemannLiouville\ fractional\ derivative).$$

Our current study considers the definition with respect to a fractional derivative operator D_ϑ^τ of $h \in \phi(R)$, then

$$(D_\vartheta^\tau h)(r) = (1\vartheta)(D_+^\tau h)(r) - \vartheta(D_-^\tau h)(r), 0 < \tau \leq 1, \vartheta \in R,$$

where D_-^τ and D_+^τ denote the left-hand and right-hand RiemannLiouville fractional derivatives of order τ, in which $0 < \tau < 1$.

We will denote the Caputo differ integral by the capital letter with upper-left index ${}^{C}D$. The fractional integral is given by the same expression as before, so for $\alpha > 0$, we have

$$ {}^{C}D_a^{-\alpha} = D_a^{-\alpha}, $$

The difference occurs for fractional derivatives. A non-integer-order derivative is again defined by the help of the fractional integral, but now we first differentiate f(t) in common sense and then go back by fractional integrating up to the required order. This idea leads to the following definition of the Caputo differ integral.

Definition 6. *Let a, T, α be real constants $(a < T), n_c = \max(0, -[-\alpha])$ and $f(t)$ a function which is integrable on $\langle a, T\rangle$ in case $n_c = 0$ and n_c-times differentiable on $\langle a, T\rangle$ except on a set of measure zero in case $n_c > 0$. Then the Caputo differintegral is defined for $t \in \langle a, T\rangle$ by formula:*

$$ {}^{C}D_a^{-\alpha} f(t) = I_a^{n_c-\alpha}\left(\frac{d^{n_c} f(t)}{dt^{n_c}}\right). $$

Remark 1. *For $\alpha > 0, \alpha \notin N_0$, then*

$$ {}^{C}D_a^{-\alpha} f(t) = \frac{1}{\Gamma(n_c - \alpha)} \int_a^t (t-\tau)^{n_c-\alpha-1} f^{n_c}(\tau) d\tau. $$

The reason why n_c in the definition of the Caputo derivative is different from n introduced in the Riemann-Liouville case is correspondence with integer-order derivatives. We cannot use n even in the Caputo definition because we would get wrong results for the k^{th} derivative of a function with zero $(k+1)^{th}$ derivative. This would be an effect of the paradox that we would need for the k^{th} derivative a $(k+1)$-times differentiable function.

Clearly, the Caputo derivative can also be written by the help of fractional integrals of the Riemann-Liouville type

$$ {}^{C}D_a^{-\alpha} f(t) = D_a^{-(n_c-\alpha)}\left(\frac{d^{n_c} f(t)}{dt^{n_c}}\right). $$

The Caputo derivative of order $\alpha = n_c$ is equal to the classical n_c^{th} derivative.

Definition 7. *Suppose that $\rho > 0, r > p, \rho, r, p \in \mathcal{R}$. Then*

$$ {}^{C}\mathcal{D}_r^{\tau} h(r) = \begin{cases} \frac{1}{\Gamma(p-\tau)} \int_0^r \frac{h^{(p)}(r)}{(r-\tau)^{\rho+1-p}}, & p-1 < \tau < p,\ p \in \mathcal{N}, \\ \frac{d^p}{dr^p} h(r), & \tau = p \in \mathcal{N}, \end{cases} $$

is called the Caputo fractional differential operator of order τ.

Definition 8. *The left and right Caputo fractional derivatives ${}_{l}^{c}\mathcal{D}_r^{\tau} h(r)$ and ${}_{r}^{c}\mathcal{D}_b^{\tau} h(r)$ of order $\tau \in \mathcal{R}_+$ are defined by*

$$ {}_{L}^{c}\mathcal{D}_a^{\tau} h(r) =_L \mathcal{D}_a^{\tau} h(r) - \sum_{k=0}^{p-1} \frac{h^k(a)}{k!}(r-a)^k \ \ (left\ Caputo\ fractional\ derivatives), $$

and

$$ {}^{c}_{R}\mathcal{D}^{\tau}_{b}h(r) ={}_R \mathcal{D}^{\tau}_{b}h(r) - \sum_{k=0}^{p-1} \frac{h^k(b)}{k!}(b-r)^k \ \ (right\ Caputo\ fractional\ derivatives), $$

respectively, where $p = \tau + 1$ *for* $\tau \in \mathcal{N}_0, p = \tau$ *for* τ. *In particular, when* $0 < \tau < 1$, *then* ${}^{c}_{L}\mathcal{D}^{\tau}_{a}h(r) ={}_L \mathcal{D}^{\tau}_{a}(h(r) - h(a))$ *and* ${}^{c}_{R}\mathcal{D}^{\tau}_{b}h(r) ={}_R \mathcal{D}^{\tau}_{b}(h(r) - h(b))$.

Remark 2. *The fractional Fourier transform and Caputo derivative are one-to-one functions.*

Some properties of fractional Fourier transform that are closely related to the solution in this study are given, as follows

(1) $\mathcal{D}^{\vartheta}_{*}f(r) = J^{\ell-\vartheta}\mathcal{D}^{\vartheta}f(r)$,
(2) $\lim_{\rho\to n}\mathcal{D}^{\vartheta}_{*}f(r) = f^{\ell}(r)$,
(3) $\mathcal{D}^{\vartheta}_{*}[\tau f(r) + g(r)] = \tau\mathcal{D}^{\vartheta}_{*}f(r) + \mathcal{D}^{\vartheta}_{*}g(r)$,
(4) $\mathcal{D}^{\vartheta}_{*}\mathcal{D}^{q}f(r) = \mathcal{D}^{\vartheta+\varrho}_{*}f(r) \neq \mathcal{D}^{q}\mathcal{D}^{\vartheta}_{*}f(r)$,
(5) $\{\mathcal{D}^{\vartheta}_{*}f(r); s\} = s^{\vartheta}F(s) - \sum_{k=0}^{\ell-1} s^{\vartheta-k-1}f^{(k)}(0)$,
(6) $\{\mathcal{D}^{\vartheta}_{*}f(r); \varpi = \left(-i\varpi^{\frac{1}{\rho}}\right)$,
(7) If $f(r) = c$ =constant, then $\mathcal{D}^{\vartheta}_{*}c = 0, c =$ constant and

$$ \mathcal{D}^{\vartheta}_{*}(f(r)g(r)) = \sum_{k=0}^{\infty}\binom{\vartheta}{k}\left(\mathcal{D}^{\vartheta-k}f(r)\right)g^{(k)}(r) - \sum_{k=0}^{\ell-1}\frac{r^{k-\vartheta}}{\Gamma(k+1-\vartheta)}\left((f(r)g(r))^{(k)}(0)\right). $$

Theorem 1. *Let* $r > 0, \tau \in \mathcal{R}, p-1 < \tau < p, p \in \mathcal{N}$. *Then the following relation between the Riemann-Lioville and the Caupto operators holds*

$$ \mathcal{D}^{\tau}_{*}h(r) = \mathcal{D}^{\tau}h(r) - \sum_{k=0}^{p-1}\frac{r^{k-\tau}}{\Gamma(k+1-\tau)}h^{(k)}(r). $$

Remark 3. *For* $n = 1$, *i.e.,* $0 < \tau < 1$ *one more* $\mathcal{D}^{\tau}_{*}r^p = \mathcal{D}^{\tau}r^p$.

Definition 9. *The MittagLeffler function can be defined in terms of a power series as*

$$ E_{\tau}(\mathfrak{c}) = \sum_{k=0}^{\infty}\frac{\mathfrak{c}^k}{\Gamma(\tau k+1)},\ \tau > 0\ (one\ parameter), \tag{2.1} $$

$$ E_{\tau,\varrho}(\mathfrak{c}) = \sum_{k=0}^{\infty}\frac{\mathfrak{c}^k}{\Gamma(\varrho+\tau k)},\ \tau > 0, \varrho > 0\ (two\ parameter). \tag{2.2} $$

Definition 10. *The fractional differential equation* $\varphi(f, y, \mathcal{D}^{\tau_1}y, \mathcal{D}^{\tau_2}y, ..., \mathcal{D}^{\tau_n}y) = 0$ *has Hyer-Ulam stability if for any continuously differentiable function y satisfies the following inequality*

$$ |\varphi(f, y, \mathcal{D}^{\tau_1}y, \mathcal{D}^{\tau_2}y, ..., \mathcal{D}^{\tau_n}y)| < \epsilon,\ \epsilon > 0, \tag{2.3} $$

then there exist a solution y_0 *of* (2.3) *such that*

$$ |y(\mathfrak{c}) - y_0(\mathfrak{c})| < \mathbb{K}(\epsilon)\ and\ \lim_{\epsilon\to 0}k(\epsilon) = 0, $$

where k is a stability constant.

Remark 4. *Let* $\varrho \in \mathbb{C}, \mathcal{R}(\varrho) > 0, r > 0$. *Then* $\mathbb{F}_{\rho}(r^{\varrho}) = \frac{\Gamma(\varrho+1)}{\left(-i\sigma^{\frac{1}{\rho}}\right)^{\varrho+1}}$.

3. Main results

In this section, we discuss the Hyer-Ulam-Mittag-Leffler stability of fractional-order linear and non-linear differential equations. Furthermore, these corollaries givens some stable results based on the following theorem and lemma.

Theorem 2. *Let* $\gamma, \delta, \in \mathbb{C}, \mathcal{R}(\gamma) > 0, \mathcal{R}(\delta) > 0, \lambda \in \mathcal{R}$. *Then*

$$\mathbb{F}_\alpha\left(r^{\gamma m+\delta-1}\mathbb{E}_\gamma^{(m)}(\lambda r^\gamma)\right) = \frac{\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\gamma-\delta} m!}{\left[\left(-i\sigma^{\frac{1}{\alpha}}\right)^\gamma - \lambda\right]^{m+1}}.$$

Lemma 1. *If* $\mathcal{R}\left(-i\sigma^{\frac{1}{\alpha}}\right) > 0, \lambda \in \mathbb{C}$ *and* $\left|\lambda\left(-i\sigma^{\frac{1}{\alpha}}\right)\right| < 1$, *then*

(1) If $\gamma = \delta = \tau, r = \mathfrak{c}, m = 0$, *then*

$$\mathbb{F}_\alpha\left(\mathfrak{c}^{\tau-1}\mathbb{E}_{\tau,\tau}(\lambda x^\tau)\right) = \frac{1}{\left[\left(-i\sigma^{\frac{1}{\alpha}}\right)^\tau - \lambda\right]}. \tag{3.1}$$

(2) If $\gamma = \tau, r = \mathfrak{c}, m = 0$, *then*

$$\mathbb{F}_\alpha\left(\mathfrak{c}^{\delta-1}\mathbb{E}_{\tau,\delta}(\lambda x^\tau)\right) = \frac{\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau-\delta}}{\left[\left(-i\sigma^{\frac{1}{\alpha}}\right)^\tau - \lambda\right]}. \tag{3.2}$$

(3) If $\gamma = \tau - \delta, r = \mathfrak{c}, m = 0$, *then*

$$\mathbb{F}_\alpha\left(\mathfrak{c}^{\tau-1}\mathbb{E}_{\tau-\delta,\tau}(\lambda x^{\tau-\delta})\right) = \frac{1}{\left[\left(-i\sigma^{\frac{1}{\alpha}}\right)^\tau - \lambda\left(-i\sigma^{\frac{1}{\alpha}}\right)^\delta\right]}. \tag{3.3}$$

3.1. Hyers-Ulam-Mittag-Leffler stability of linear fractional differential equation

In this part, we are going to analyse the Hyers-Ulam-Mittag-Leffler stability of the linear fractional differential equation of the form

$$\left({}^c\mathcal{D}_{0+}^\alpha y\right)(\mathfrak{c}) - \lambda\left({}^c\mathcal{D}_{0+}^\delta y\right)y(\mathfrak{c}) = h(\mathfrak{c}),$$

by using fractional Fourier transform method.

Theorem 3. *Let* $\lambda \in \mathcal{R}, p-1 < \tau \leq p, p \in \mathcal{N}$ *and let* $h(\mathfrak{c})$ *be a real valued function defined on* $\mathcal{R}$. *If a function* $y : (0,\infty) \to \mathcal{R}$ *satisfies*

$$\left|\left({}^c\mathcal{D}_{0+}^\alpha y\right)(\mathfrak{c}) - \lambda\left({}^c\mathcal{D}_{0+}^\delta y\right)y(\mathfrak{c}) - h(\mathfrak{c})\right| \leq \epsilon, \epsilon > 0, \ \forall\ x > 0, \tag{3.4}$$

then there exists a solution $y_a : (0,\infty) \to \mathcal{R}$ *of* $\left({}^c\mathcal{D}_{0+}^\alpha y\right)(\mathfrak{c}) - \lambda\left({}^c\mathcal{D}_{0+}^\delta y\right)y(\mathfrak{c}) = h(\mathfrak{c})$ *such that*

$$|y - y_a| \leq \epsilon\mathfrak{c}^\tau\mathbb{E}_{\tau,\tau+1}\left(|\lambda|\mathfrak{c}^{\tau-\delta}\right). \tag{3.5}$$

Proof. Putting $y^{(k)}(0) = b_k$, for $k = 0, 1, 2, ..., p-1$ and

$$y(\mathfrak{c}) = ({}^c\mathcal{D}^{\alpha}_{0+}y)(\mathfrak{c}) - \lambda\left({}^c\mathcal{D}^{\delta}_{0+}y\right)y(\mathfrak{c}) - h(\mathfrak{c}).$$

Now,

$$y(\mathfrak{c}) = \left[\mathcal{D}^{\tau}y(\mathfrak{c}) - \sum_{k=0}^{p-1}\frac{\mathfrak{c}^{k-\tau}}{\Gamma(k+1-\tau)}y^k(0)\right] - \lambda\left[\mathcal{D}^{\tau}y(\mathfrak{c}) - \sum_{k=0}^{p-1}\frac{\mathfrak{c}^{k-\delta}}{\Gamma(k+1-\tau)}y^k(0)\right] - h(\mathfrak{c}).$$

Taking fractional Fourier transform on both sides, we have

$$\begin{aligned}\mathbb{F}_{\alpha}[Y(\mathfrak{c})] &= \mathbb{F}_{\alpha}[\mathcal{D}^{\tau}y(\mathfrak{c})] - \sum_{k=0}^{p-1}\frac{b_k}{\Gamma(k+1-\tau)}\mathbb{F}_{\alpha}\left[\mathfrak{c}^{k-\tau}\right] - \lambda\left[\mathbb{F}_{\alpha}[\mathcal{D}^{\tau}y(\mathfrak{c})]\sum_{k=0}^{p-1}\frac{b_k}{\Gamma(k+1-\delta)}\mathbb{F}_{\alpha}\left[\mathfrak{c}^{k-\delta}\right]\right] - \mathbb{F}_{\alpha}[h(\mathfrak{c})] \qquad (3.6)\\ &= \left[\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau} - \lambda\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\delta}\right]\mathbb{F}_{\alpha}[y(\mathfrak{c})] - \mathbb{F}_{\alpha}[h(\mathfrak{c})] - \sum_{k=0}^{p-1}b_k\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau-k-1} + \lambda\sum_{k=0}^{p-1}b_k\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\delta-k-1},\end{aligned}$$

Which implies that

$$\left[\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau} - \lambda\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\delta}\right]\mathbb{F}_{\alpha}[y(\mathfrak{c})] = \mathbb{F}_{\alpha}[Y(\mathfrak{c})] - \mathbb{F}_{\alpha}[h(\mathfrak{c})] + \sum_{k=0}^{p-1}b_k\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau-k-1} - \lambda\sum_{k=0}^{p-1}b_k\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\delta-k-1}$$

$$\mathbb{F}_{\alpha}[y(\mathfrak{c})] = \frac{\mathbb{F}_{\alpha}[Y(\mathfrak{c})] - \mathbb{F}_{\alpha}[h(\mathfrak{c})]}{\left[\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau} - \lambda\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\delta}\right]} + \frac{\sum_{k=0}^{p-1}b_k\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau-k-1}}{\left[\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau} - \lambda\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\delta}\right]} + \lambda\frac{\sum_{k=0}^{p-1}b_k\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau-k-1}}{\left[\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau} - \lambda\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\delta}\right]}. \qquad (3.7)$$

Setting

$$y_0(\mathfrak{c}) = \sum_{k=0}^{p-1}b_ky_k(\mathfrak{c}) + \sum_{k=n}^{q-1}b_ky_k(\mathfrak{c}) + \int_0^{\mathfrak{c}}(\mathfrak{c}-r)^{\tau-1}\mathbb{E}_{\tau-\delta,\tau}\left[\lambda(\mathfrak{c}-r)^{\tau-\delta}\right]h(r)dr, \qquad (3.8)$$

where $y_k(\mathfrak{c}) = \mathfrak{c}^k\mathbb{E}_{\tau-\delta,k+1}(\lambda\mathfrak{c}^{\tau-\delta}) - \lambda\mathfrak{c}^{\tau-\delta+k}\mathbb{E}_{\tau-\delta,\tau-\delta+k+1}(\lambda\mathfrak{c}^{\tau-\delta}), k = 0, 1, 2, ..., q-1$,

$y_k(\mathfrak{c}) = \mathfrak{c}^k\mathbb{E}_{\tau-\delta,k+1}(\lambda\mathfrak{c}^{\tau-\delta}), k = q, ..., p-1$,
and

$$y_0(\mathfrak{c}) = \sum_{k=n}^{q-1}b_ky_k(\mathfrak{c}) + \int_0^{\mathfrak{c}}(\mathfrak{c}-r)^{\tau-1}\mathbb{E}_{\tau-\delta,\tau}\left[\lambda(\mathfrak{c}-r)^{\tau-\delta}\right]h(r)dr.$$

Taking fractional Fourier transform on both sides, we have

$$\begin{aligned}\mathbb{F}_{\alpha}[y_a(\mathfrak{c})] &= \sum_{k=0}^{p-1}b_k\mathbb{F}_{\alpha}\left[\mathfrak{c}^k\mathbb{E}_{\tau-\delta,k+1}\left(\lambda\mathfrak{c}^{\tau-\delta}\right) - \lambda\mathfrak{c}^{\tau-\delta+k}\mathbb{E}_{\tau-\delta,\tau-\delta+K+1}\left(\lambda\mathfrak{c}^{\tau+\delta}\right)\right] + \sum_{k=0}^{p-1}b_k\mathbb{F}_{\alpha}\left[\mathfrak{c}^k\mathbb{E}_{\tau-\delta,k+1}\left(\lambda\mathfrak{c}^{\tau-\delta}\right)\right]\\ &+ \mathbb{F}_{\alpha}\left[\int_0^{\mathfrak{c}}(\mathfrak{c}-r)^{\tau-1}\mathbb{E}_{\tau-\delta,\tau}\left[\lambda(\mathfrak{c}-r)^{\tau-\delta}\right]\right]\mathbb{F}_{\alpha}[h(r)]\,dr.\end{aligned}$$

Consequently,

$$\mathbb{F}_\alpha\left[y_a(\mathfrak{c})\right] = \frac{\left[\sum_{k=0}^{p-1} b_k\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau-k-1} - \lambda\sum_{k=0}^{p-1} b_k\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\delta-k-1} + \mathbb{F}_\alpha(h(\mathfrak{c}))\right]}{\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau} - \lambda\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\delta}}. \tag{3.9}$$

By (3.7) and a simple computation, we get

$$\left({}^{c}\mathcal{D}_{0+}^{\alpha}y_a\right)(\mathfrak{c}) - \lambda\left({}^{c}\mathcal{D}_{0+}^{\delta}y_a\right)y_a(\mathfrak{c}) = \left[\mathcal{D}^{\tau}y_a(\mathfrak{c}) - \sum_{k=0}^{p-1}\frac{\mathfrak{c}^{k-\tau}}{\Gamma(k+1-\tau)}y^{(k)}(0)\right] - \lambda\left[\mathcal{D}^{\delta}y_a(\mathfrak{c}) - \sum_{k=0}^{p-1}\frac{\mathfrak{c}^{k-\delta}}{\Gamma(k+1-\delta)}y^{(k)}(0)\right]$$

$$\begin{aligned}\mathbb{F}_\alpha\left[\left({}^{c}\mathcal{D}_{0+}^{\alpha}y_a\right)(\mathfrak{c}) - \lambda\left({}^{c}\mathcal{D}_{0+}^{\delta}y_a\right)(\mathfrak{c})\right] &= \left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau}\mathbb{F}_\alpha\left[y_a(\mathfrak{c})\right]\sum_{k=0}^{p-1} b_k\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau-k-1}\\ &\quad - \lambda\left(\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau}\mathbb{F}_\alpha\left[y_a(\mathfrak{c})\right]\right) + \lambda\sum_{k=0}^{p-1} b_k\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\delta-k-1}\\ &= \mathbb{F}_\alpha\left[h(\mathfrak{c})\right].\end{aligned} \tag{3.10}$$

Since $\mathbb{F}_\alpha$ is 1-1, it follows that

$$\left({}^{c}\mathcal{D}_{0+}^{\alpha}y_a\right)(\mathfrak{c}) - \lambda y(\mathfrak{c}) = h(\mathfrak{c}).$$

So $y_0(\mathfrak{c})$ is a solution of (3.4). By (3.7) and (3.9), we get

$$\mathbb{F}_\alpha\left(y(\mathfrak{c}) - y_a(\mathfrak{c})\right) = \mathbb{F}_\alpha\left(y(\mathfrak{c})\right) - \mathbb{F}_\alpha\left(y_a(\mathfrak{c})\right) = \frac{\mathbb{F}_\alpha\left(Y(\mathfrak{c})\right)}{\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau} - \lambda\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\delta}}. \tag{3.11}$$

Using the convolution property, we obtain

$$\mathbb{F}_\alpha\left(\mathfrak{c}^{\tau-1}\mathbb{E}_{\tau-\delta,\alpha}(\lambda\mathfrak{c}^{\tau-\delta}) * Y(\mathfrak{c})\right) = \mathbb{F}_\alpha\left(\mathfrak{c}^{\tau-1}\mathbb{E}_{\tau-\delta,\tau}(\lambda\mathfrak{c}^{\tau-\delta})\right)\mathbb{F}_\alpha\left(Y(\mathfrak{c})\right) = \frac{\mathbb{F}_\alpha\left(Y(\mathfrak{c})\right)}{\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau} - \lambda\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\delta}}. \tag{3.12}$$

By (3.11) and (3.12), we have

$$y(\mathfrak{c}) - y_a(\mathfrak{c}) = \left(\mathfrak{c}^{\tau-1}\mathbb{E}_{\tau-\delta,\tau}(\lambda\mathfrak{c}^{\alpha-\delta})\right) * Y(\mathfrak{c}). \tag{3.13}$$

Therefore, from (3.3) it follows that

$$\begin{aligned}|y(\mathfrak{c}) - y_a(\mathfrak{c})| &= \left|\left(\mathfrak{c}^{\tau-1}\mathbb{E}_{\tau-\delta,\tau}(\lambda\mathfrak{c}^{\alpha-\delta})\right) * Y(\mathfrak{c})\right|\\ &= \left|\int_0^{\mathfrak{c}}(\mathfrak{c}-r)^{\tau-1}\mathbb{E}_{\tau-\delta,\tau}(\lambda(\mathfrak{c}-r)^{\tau-\delta})Y(r)dr\right|\\ &= \epsilon\mathfrak{c}^{\tau}\mathbb{E}_{\tau-\delta,\tau+1}(|\lambda|\mathfrak{c}^{\tau-\delta}).\end{aligned} \tag{3.14}$$

Then by definition of Hyers-Ulam stability, (3.3) has the Hyers-Ulam stability. □

Corollary 1. *Let $\lambda \in \mathcal{R}$, $p-1 < \tau \leq p$, $p \in \mathcal{N}$ and let $h(\mathfrak{c})$ be a real valued function defined on $\mathcal{R}$, also $\chi(\mathfrak{c}) \in \mathcal{R}$. If $y_0 : (0,\infty) \to \mathcal{R}$ satisfies*

$$\left| \left({}^{c}\mathcal{D}_{0+}^{\tau} y \right)(\mathfrak{c}) - \lambda \left({}^{c}\mathcal{D}_{0+}^{\delta} \right) y(\mathfrak{c}) - h(\mathfrak{c}) \right| \leq \chi(\mathfrak{c})\epsilon, \ \epsilon > 0, \ \forall \ x > 0, \tag{3.15}$$

then there exists a solution $y_0 : (0,\infty) \to \mathcal{R}$ of

$$\left({}^{c}\mathcal{D}_{0+}^{\tau} y \right)(\mathfrak{c}) - \lambda \left({}^{c}\mathcal{D}_{0+}^{\delta} \right) y(\mathfrak{c}) = h(\mathfrak{c}),$$

such that

$$|y - y_0| \leq \epsilon \chi(\mathfrak{c}) \mathfrak{c}^{\tau} \mathbb{E}_{\tau-\delta,\tau+1} \left(|\lambda| \mathfrak{c}^{\tau} \right). \tag{3.16}$$

3.2. Hyers-Ulam stability of the non-linear fractional differential equation

In this section, we are going to analyse the Hyers-Ulam-Mittag-Leffler stability of the non-Linear fractional differential equation of the form

$$\left({}^{c}\mathcal{D}_{0+}^{\alpha} y \right)(\mathfrak{c}) - \lambda \left({}^{c}\mathcal{D}_{0+}^{\delta} \right) y(\mathfrak{c}) - h(\mathfrak{c}) = F(\mathfrak{c}),$$

by using the fractional Fourier transform method.

Theorem 4. *Let $\lambda \in \mathcal{R}$, $p-1 < \tau \leq p$, $p \in \mathcal{N}$ and let $h(\mathfrak{c})$ be a real valued function defined on $\mathcal{R}$. If a function $y : (0,\infty) \to \mathcal{R}$ satisfies*

$$\left| \left({}^{c}\mathcal{D}_{0+}^{\alpha} y \right)(\mathfrak{c}) - \lambda \left({}^{c}\mathcal{D}_{0+}^{\delta} \right) y(\mathfrak{c}) - h(\mathfrak{c}) \right| \leq F(\mathfrak{c}), \epsilon > 0, \ \forall \ x > 0, \tag{3.17}$$

then there exists a solution $y_a : (0,\infty) \to \mathcal{R}$ of

$$\left({}^{c}\mathcal{D}_{0+}^{\alpha} y \right)(\mathfrak{c}) - \lambda \left({}^{c}\mathcal{D}_{0+}^{\delta} \right) y(\mathfrak{c}) - h(\mathfrak{c}) = F(\mathfrak{c}),$$

such that

$$|y - y_a| \leq C(\mathfrak{c}), \ \text{here } C(\mathfrak{c}) = \mathfrak{c}^{\tau} \mathbb{E}_{\tau-\delta,\tau+1} \left(|\lambda| \mathfrak{c}^{\tau-\delta} \right). \tag{3.18}$$

Proof. Putting $y^{(k)}(0) = b_k$, for $k = 0, 1, 2, ..., p-1$ and

$$y(\mathfrak{c}) = \left({}^{c}\mathcal{D}_{0+}^{\alpha} y \right)(\mathfrak{c}) - \lambda \left({}^{c}\mathcal{D}_{0+}^{\delta} \right) y(\mathfrak{c}) - h(\mathfrak{c}) - F(\mathfrak{c}),$$

$$y(\mathfrak{c}) = \left[\mathcal{D}^{\tau} y(\mathfrak{c}) - \sum_{k=0}^{p-1} \frac{\mathfrak{c}^{k-\tau}}{\Gamma(k+1-\tau)} y^{k}(0) \right] - \lambda \left[\mathcal{D}^{\delta} y(\mathfrak{c}) - \sum_{k=0}^{p-1} \frac{\mathfrak{c}^{k-\delta}}{\Gamma(k+1-\delta)} y^{k}(0) \right] - h(\mathfrak{c}) - F(\mathfrak{c}).$$

Taking fractional Fourier transform on both sides, we have

$$\mathbb{F}_{\alpha}\left[y(\mathfrak{c})\right] = \mathbb{F}_{\alpha}\left[\mathcal{D}^{\tau} y(\mathfrak{c})\right] - \sum_{k=0}^{p-1} \frac{b_k}{\Gamma(k+1-\tau)} \mathbb{F}_{\alpha}\left[\mathfrak{c}^{k-\tau}\right] - \lambda \left[\mathbb{F}_{\alpha}\left[\mathcal{D}^{\tau} y(\mathfrak{c})\right] - \sum_{k=0}^{p-1} \frac{b_k}{\Gamma(k+1-\tau)} \mathbb{F}_{\alpha}\left[\mathfrak{c}^{k-\tau}\right] \right] - \mathbb{F}_{\alpha}\left[h(\mathfrak{c})\right] - \mathbb{F}_{\alpha}\left[F(\mathfrak{c})\right] \tag{3.19}$$

$$= \left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau} \mathbb{F}_{\alpha}\left[y(\mathfrak{c})\right] - \sum_{k=0}^{p-1} \frac{b_k}{\Gamma(k+1-\tau)} \frac{\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau-k-1}}{\Gamma(k+1-\tau)} - \lambda \left[\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau} \mathbb{F}_{\alpha}\left[y(\mathfrak{c})\right] - \sum_{k=0}^{p-1} \frac{b_k}{\Gamma(k+1-\tau)} \frac{\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau-k-1}}{\Gamma(k+1-\tau)} \right] - \mathbb{F}_{\alpha}\left[h(\mathfrak{c})\right]. \tag{3.20}$$

That is,

$$\left[\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau}-\lambda\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\delta}\right]\mathbb{F}_{\alpha}\left[y(\mathfrak{c})\right]=\sum_{k=0}^{p-1}b_{k}\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau-k-1}-\lambda\sum_{k=0}^{p-1}b_{k}\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\delta-k-1}+\mathbb{F}_{\alpha}\left[Y(\mathfrak{c})\right]+\mathbb{F}_{\alpha}\left[h(\mathfrak{c})\right]+\mathbb{F}_{\alpha}\left[F(\mathfrak{c})\right],$$

$$\mathbb{F}_{\alpha}\left[y(\mathfrak{c})\right]=\frac{\sum_{k=0}^{p-1}b_{k}\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau-k-1}}{\left[\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau}-\lambda\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\delta}\right]}-\frac{\lambda\sum_{k=0}^{p-1}b_{k}\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\delta-k-1}}{\left[\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau}-\lambda\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\delta}\right]}+\frac{\mathbb{F}_{\alpha}\left[Y(\mathfrak{c})\right]+\mathbb{F}_{\alpha}\left[h(\mathfrak{c})\right]+\mathbb{F}_{\alpha}\left[F(\mathfrak{c})\right]}{\left[\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau}-\lambda\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\delta}\right]}. \tag{3.21}$$

Setting

$$\begin{aligned}y_{0}(\mathfrak{c})=&\sum_{k=0}^{q-1}b_{k}y_{k}(\mathfrak{c})\sum_{k=m}^{p-1}b_{k}y_{k}(\mathfrak{c})+\int_{0}^{\mathfrak{c}}(\mathfrak{c}-r)^{\tau-1}\mathbb{E}_{\tau-\delta,\tau}\left[\lambda(\mathfrak{c}-r)^{\tau-\delta}\right]h(r)dr\\&+\int_{0}^{\mathfrak{c}}(\mathfrak{c}-r)^{\tau-1}\mathbb{E}_{\tau-\delta,\tau}\left[\lambda(\mathfrak{c}-r)^{\tau-\delta}\right]F(r)dr,\end{aligned} \tag{3.22}$$

where $y_{k}(\mathfrak{c})=\mathfrak{c}^{k}\mathbb{E}_{\tau-\delta,K+1}(\lambda\mathfrak{c}^{\tau-\delta})-\lambda\mathfrak{c}^{\tau-\delta+k}\mathbb{E}_{\tau-\delta,\tau-\delta+k+1}(\lambda\mathfrak{c}^{\tau-\delta}),k=0,1,2,...,q-1,$

$y_{k}(\mathfrak{c})=\mathfrak{c}^{k}\mathbb{E}_{\tau-\delta,K+1}(\lambda\mathfrak{c}^{\tau-\delta}),k=q,...,p-1,$
and

$$y_{a}(\mathfrak{c})=\sum_{k=1}^{p-1}b_{k}y_{k}(\mathfrak{c})+\int_{0}^{\mathfrak{c}}(\mathfrak{c}-r)^{\tau-1}\mathbb{E}_{\tau-\delta,\tau}\left[\lambda(\mathfrak{c}-r)^{\tau-\delta}\right]h(r)dr+\int_{0}^{\mathfrak{c}}(\mathfrak{c}-r)^{\tau-1}\mathbb{E}_{\tau-\delta,\tau}\left[\lambda(\mathfrak{c}-r)^{\tau-\delta}\right]F(r)dr.$$

Taking fractional Fourier transform on both sides, we have

$$\begin{aligned}\mathbb{F}_{\alpha}\left[y_{a}(\mathfrak{c})\right]=&\sum_{k=1}^{q-1}b_{k}\mathbb{F}_{\alpha}\left[\mathfrak{c}^{k}\mathbb{E}_{\tau-\delta,k+1}\left(\lambda\mathfrak{c}^{\tau-\delta}\right)-\lambda\mathfrak{c}^{\tau-\delta+k}\mathbb{E}_{\tau-\delta,\tau-\delta+k+1}\left(\lambda\mathfrak{c}^{\tau+\delta}\right)\right]+\sum_{k=m}^{p-1}b_{k}\mathbb{F}_{\alpha}\left[\mathfrak{c}^{k}\mathbb{E}_{\tau-\delta,k+1}\left(\lambda\mathfrak{c}^{\tau-\delta}\right)\right]\\&+\mathbb{F}_{\alpha}\left[\int_{0}^{\mathfrak{c}}(\mathfrak{c}-r)^{\tau-1}\mathbb{E}_{\tau-\delta,\tau}\left[\lambda(\mathfrak{c}-r)^{\tau-\delta}\right]\right]\mathbb{F}_{\alpha}\left[h(r)\right]dr+\mathbb{F}_{\alpha}\left[\int_{0}^{\mathfrak{c}}(\mathfrak{c}-r)^{\tau-1}\mathbb{E}_{\tau-\delta,\tau}\left[\lambda(\mathfrak{c}-r)^{\tau-\delta}\right]\right]\mathbb{F}_{\alpha}\left[F(r)\right]dr.\end{aligned}$$

That is,

$$\mathbb{F}_{\alpha}\left[y_{a}(\mathfrak{c})\right]=\frac{\left[\sum_{k=0}^{p-1}b_{k}\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau-k-1}-\lambda\sum_{k=0}^{p-1}b_{k}\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\delta-k-1}+\mathbb{F}_{\alpha}(h(\mathfrak{c}))++\mathbb{F}_{\alpha}(F(\mathfrak{c}))\right]}{\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau}-\lambda\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\delta}}. \tag{3.23}$$

By (3.7) and a simple computation, we get

$$\left({}^{c}\mathcal{D}_{0+}^{\alpha}y_{a}\right)(\mathfrak{c})-\lambda\left({}^{c}\mathcal{D}_{0+}^{\delta}y_{a}\right)(\mathfrak{c})-h(\mathfrak{c})=\left[\mathcal{D}^{\tau}y_{0}(\mathfrak{c})-\sum_{k=0}^{p-1}\frac{\mathfrak{c}^{k-\tau}}{\Gamma(k+1-\tau)}y_{a}^{(k)}(0)\right]-\lambda\left[\mathcal{D}^{\delta}y_{a}(\mathfrak{c})-\sum_{k=0}^{p-1}\frac{\mathfrak{c}^{k-\delta}}{\Gamma(k+1-\delta)}y_{a}^{(k)}(0)\right]-h(\mathfrak{c}),$$

$$\mathbb{F}_{\alpha}\left[\left({}^{c}\mathcal{D}_{0+}^{\alpha}y_{a}\right)(\mathfrak{c})-\lambda\left({}^{c}\mathcal{D}_{0+}^{\delta}y_{a}\right)(\mathfrak{c})\right]=\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau}\mathbb{F}_{\alpha}\left[y_{a}(\mathfrak{c})\right]\sum_{k=0}^{p-1}b_{k}\left(-i\sigma^{\frac{1}{\alpha}}\right)^{\tau-k-1}$$

$$- \lambda \left(\left(-i\sigma^{\frac{1}{\alpha}} \right)^{\tau} \mathbb{F}_{\alpha} \left[y_a(\mathfrak{c}) \right] \right) + \lambda \sum_{k=0}^{p-1} b_k \left(-i\sigma^{\frac{1}{\alpha}} \right)^{\delta-k-1} - \mathbb{F}_{\alpha} \left[h(\mathfrak{c}) \right]$$
$$= \mathbb{F}_{\alpha} \left[h(\mathfrak{c}) \right]. \tag{3.24}$$

Since $\mathbb{F}_{\alpha}$ is 1-1, it follows that

$$\left({}^{c}\mathcal{D}_{0+}^{\alpha} y_a \right)(\mathfrak{c}) - \lambda y(\mathfrak{c}) = h(\mathfrak{c}).$$

So $y_a(\mathfrak{c})$ is a solution of (3.4). By (3.7) and (3.9), we get

$$\mathbb{F}_{\alpha} \left(y(\mathfrak{c}) - y_0(\mathfrak{c}) \right) = \mathbb{F}_{\alpha} \left(y(\mathfrak{c}) \right) - \mathbb{F}_{\alpha} \left(y_a(\mathfrak{c}) \right) = \frac{\mathbb{F}_{\alpha} \left(Y(\mathfrak{c}) \right)}{\left(-i\sigma^{\frac{1}{\alpha}} \right)^{\tau} - \lambda \left(-i\sigma^{\frac{1}{\alpha}} \right)^{\delta}}. \tag{3.25}$$

Using the convolution property, we can get

$$\mathbb{F}_{\alpha} \left(\mathfrak{c}^{\tau-1} \mathbb{E}_{\tau-\delta,\alpha}(\lambda \mathfrak{c}^{\tau-\delta}) * Y(\mathfrak{c}) \right) = \mathbb{F}_{\alpha} \left(\mathfrak{c}^{\tau-1} \mathbb{E}_{\tau-\delta,\tau}(\lambda \mathfrak{c}^{\tau-\delta}) \right) \mathbb{F}_{\alpha} \left(Y(\mathfrak{c}) \right) = \frac{\mathbb{F}_{\alpha} \left(Y(\mathfrak{c}) \right)}{\left(-i\sigma^{\frac{1}{\alpha}} \right)^{\tau} - \lambda \left(-i\sigma^{\frac{1}{\alpha}} \right)^{\delta}}. \tag{3.26}$$

By (3.11) and (3.12), we have

$$y(\mathfrak{c}) - y_a(\mathfrak{c}) = \left(\mathfrak{c}^{\tau-1} \mathbb{E}_{\alpha,\alpha+1}(\lambda \mathfrak{c}^{\alpha-\delta}) \right) * Y(\mathfrak{c}). \tag{3.27}$$

Therefore, from (3.3) it follows that

$$\begin{aligned} |y(\mathfrak{c}) - y_a(\mathfrak{c})| &= \left| \left(\mathfrak{c}^{\tau-1} \mathbb{E}_{\tau-\delta,\tau}(\lambda \mathfrak{c}^{\alpha-\delta}) \right) * Y(\mathfrak{c}) \right| \\ &= \left| \int_0^{\mathfrak{c}} (\mathfrak{c} - r)^{\tau-1} \mathbb{E}_{\tau-\delta,\tau}(\lambda (\mathfrak{c} - r)^{\tau-\delta}) Y(r) dr \right| \\ &= F(\mathfrak{c}) \mathfrak{c}^{\tau} \mathbb{E}_{\tau-\delta,\tau+1}(|\lambda| \mathfrak{c}^{\tau-\delta}). \end{aligned} \tag{3.28}$$

Then by definition of Hyers-Ulam-Mittag-Leffler stability, the fractional differential Eq (1.2) has the Hyers-Ulam stability. □

Corollary 2. *Let $\lambda \in \mathcal{R}$, $p - 1 < \tau \leq p$, $p \in \mathcal{N}$ and let $h(\mathfrak{c})$ be a real valued function defined on $\mathcal{R}$, also $\chi(\mathfrak{c}) \in \mathcal{R}$. If $y_a : (0, \infty) \to \mathcal{R}$ satisfies the inequality*

$$\left| \left({}^{c}\mathcal{D}_{0+}^{\tau} y \right)(\mathfrak{c}) - \lambda \left({}^{c}\mathcal{D}_{0+}^{\delta} y \right) y(\mathfrak{c}) - h(\mathfrak{c}) - h(\mathfrak{c}) \right| \leq \chi(\mathfrak{c})\epsilon, \epsilon > 0, \ \forall \ x > 0, \tag{3.29}$$

then there exists a solution $y_0 : (0, \infty) \to \mathcal{R}$ of

$$\left({}^{c}\mathcal{D}_{0+}^{\tau} y \right)(\mathfrak{c}) - \lambda \left({}^{c}\mathcal{D}_{0+}^{\delta} y \right) y(\mathfrak{c}) - h(\mathfrak{c}) = h(\mathfrak{c}),$$

such that $|y - y_0| \leq C(\mathfrak{c})$, here

$$C(\mathfrak{c}) = \chi(\mathfrak{c}) h(\mathfrak{c}) \mathfrak{c}^{\tau} \mathbb{E}_{\tau-\delta,\tau+1} \left(|\lambda| \mathfrak{c}^{\tau-\delta} \right). \tag{3.30}$$

4. Applications

We consider the fractional differential equation

$$ {}_rD^{\alpha}_{1/2}u(r,s) = {}_t\, D^{\beta}u(r,s), r \in R, s \in R_+, $$

where the α, β are real parameters always restricted as follows $0 < \alpha \le 1, 0 < \beta \le 2$, ${}_rD^{\alpha}_{1/2} = \frac{1}{2}({}_rD^{\alpha}_{+} - {}_r D^{\alpha}_{-})$ is the space-fractional derivative of order α and ${}_sD^{\beta}_{*}$ is the Caputo time-fractional derivative of order $\beta (m-1 < \beta \le m, m \in N)$ defined as follows:

$$ {}_rD^{\beta}_{*}h(s) = \begin{cases} \frac{\Gamma(m-\beta)}{\int} {}^{s}_{0} \frac{f(m)(\tau)}{(s-\tau)^{\beta+1-m}} d\tau, & m-1 < \beta < m, \\ \frac{d^m}{ds^m} h(s), & \beta = m. \end{cases} $$

This operator has been referred to as the Caputo fractional derivative since it was introduced by Caputo in the late 1960s for modeling the energy dissipation in some anelastic materials with it is well known that for a sufficiently well-behaved function h the property

$$ L\{{}_sD^{\beta}_{*}h(t); t\} = t^{\beta}\tilde{h}(t) - \sum_{k=0}^{m-1} t^{\beta-1-k} h(k)(0^+), m-1 < \beta \le m, $$

holds true, L being the Laplace transform

$$ \tilde{h}(t) = L\{h(s); t\} = \int_{o}^{\infty} e^{-ts} h(t) dt,\ R(t) > a_h, $$

of a function h. A sufficient condition of the existence of the Laplace transform is that the original function is of exponential order as $t \to \infty$. This means that some constant a_h exists such that the product $e^{-a_h s}|h(s)|$ is bounded for all t greater than some T. Then $\tilde{h}(t)$ exists and is analytic in the half plane $R(t) > a_h$.

5. Examples

In this part, some examples are given to illustrate linear fractional differential equation and non linear fractional differential equation for use our main theoretical part.

Example 1. *Let the linear fractional differential equation*

$$ \left({}^c\mathcal{D}^{\frac{1}{2}}_{0+}y\right)(\mathfrak{c}) - \frac{1}{3}\left({}^c\mathcal{D}^{\frac{1}{3}}_{0+}y\right)(\mathfrak{c}) = \frac{2}{3}\mathfrak{c}^{\frac{3}{2}} - \frac{3}{5}\frac{\mathfrak{c}^{\frac{5}{3}}}{\Gamma(\frac{5}{3})}, \tag{5.1} $$

where $\tau = \frac{1}{2}, \lambda = \frac{1}{3}, \delta = \frac{1}{3}, h(\mathfrak{c}) = \frac{2}{3}\mathfrak{c}^{\frac{3}{2}} - \frac{3}{5}\frac{\mathfrak{c}^{\frac{5}{3}}}{\Gamma(\frac{5}{3})}$.

For $\epsilon = \frac{1}{2}$, *it is very easy to check that the function* $y_1(\mathfrak{c}) = \mathfrak{c}^2$ *satisfies*

$$ \left|\left({}^c\mathcal{D}^{\frac{1}{2}}_{0+}y\right)(\mathfrak{c}) - \frac{1}{3}\left({}^c\mathcal{D}^{\frac{1}{3}}_{0+}y\right)(\mathfrak{c}) - \frac{2}{3}\mathfrak{c}^{\frac{3}{2}} + \frac{3}{5}\frac{\mathfrak{c}^{\frac{5}{3}}}{\Gamma(\frac{5}{3})}\right| < \frac{1}{2}, $$

and initial values of $y_1(\mathfrak{c})$ *are* $y_1(0) = y_1' = 0$. *From* (3.8) *and the initial values of* $y_1(\mathfrak{c})$*, we get an exact solution of Eq* (5.1)

$$y_0(\mathfrak{c}) = \int_0^{\mathfrak{c}} (\mathfrak{c}-r)^{\frac{-1}{2}} \mathbb{E}_{\frac{1}{6}\frac{1}{2}}\left(\frac{1}{3}(\mathfrak{c}-r)^{\frac{1}{6}}\right)\left(\frac{2}{3}r^{\frac{3}{2}} - \frac{3}{5}\frac{\mathfrak{c}^{\frac{5}{3}}}{\Gamma(\frac{5}{3})}\right)dr.$$

By theorem 3.3, the control function of $y_1(\mathfrak{c})$ *is* $\frac{1}{2}\mathfrak{c}^{\frac{1}{2}}\mathbb{E}_{\frac{1}{6},\frac{3}{2}}\left(\frac{1}{3}\mathfrak{c}^{\frac{1}{2}}\right)$*, thus*

$$|y_1(\mathfrak{c}) - y_0(\mathfrak{c})| < \frac{1}{2}\mathfrak{c}^{\frac{1}{2}}\mathbb{E}_{\frac{1}{6},\frac{3}{2}}\left(\frac{1}{3}\mathfrak{c}^{\frac{1}{2}}\right),$$

Using MATLAB, the solution of (5.1) *is computed and depicted in Figure 1. In addition, the error of the approximate solution* $y_1(\mathfrak{c})$ *can be estimated.*

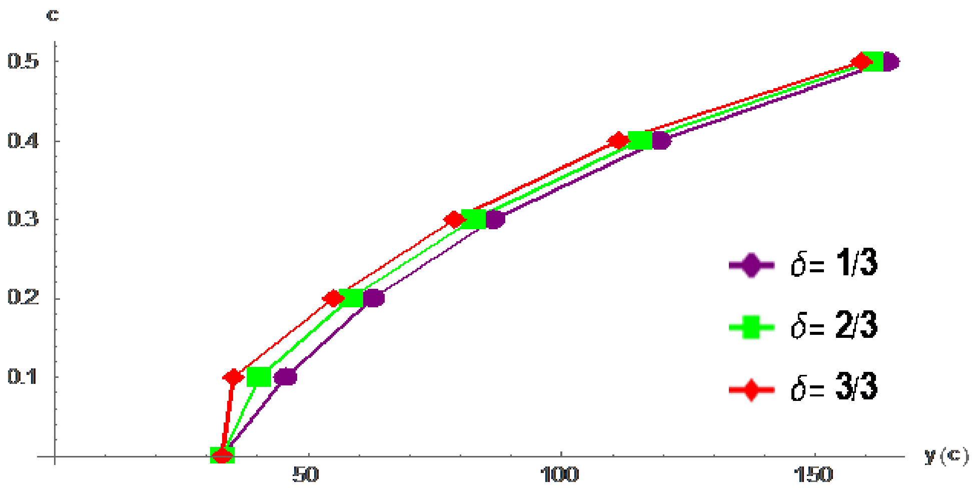

Figure 1. Solution of Eq (5.1).

Example 2. *Let the non-linear fractional differential equation*

$$\left({}^{c}\mathcal{D}^{2}_{0+}y\right)(\mathfrak{c}) - \frac{1}{3}\left({}^{c}\mathcal{D}^{\frac{5}{2}}_{0+}y\right)(\mathfrak{c}) = \frac{5}{2} - \frac{2}{3\sqrt{\pi}\sqrt{\mathfrak{c}}}, \tag{5.2}$$

where $\tau = \frac{1}{2}, \delta = \frac{5}{2}, \lambda = \frac{1}{3}, h(\mathfrak{c}) = \frac{5}{2} - \frac{2}{3\sqrt{\pi}\sqrt{\mathfrak{c}}}$.

For $\epsilon = \frac{1}{2}$*, it is very easy to check that the function* $y_1(\mathfrak{c}) = \mathfrak{c}^2$ *satisfies*

$$\left|\left({}^{c}\mathcal{D}^{2}_{0+}y\right)(\mathfrak{c}) - \frac{1}{3}\left({}^{c}\mathcal{D}^{\frac{5}{2}}_{0+}y\right)(\mathfrak{c}) - \frac{5}{2} + \frac{2}{3\sqrt{\pi}\sqrt{\mathfrak{c}}}\right| < \frac{1}{2},$$

and initial values of $y_1(\mathfrak{c})$ *and* $y_1(0)$ *are 0. From* (3.25) *and the initial values of* $y_1(\mathfrak{c})$*, we get an exact solution of Eq* (5.2)

$$y_0(\mathfrak{c}) = \int_0^{\mathfrak{c}} (\mathfrak{c}-r)^{\frac{-1}{2}} \mathbb{E}_{\frac{11}{2}\frac{1}{2}}\left(\frac{1}{3}(\mathfrak{c}-r)^{\frac{-1}{2}}\right)\left(\frac{5}{2} - \frac{2}{3\sqrt{\pi}\sqrt{\mathfrak{c}}}\right)dr.$$

By theorem 3.3, the control function of $y_1(\mathfrak{c})$ *is* $\frac{1}{2}\mathfrak{c}^2\mathbb{E}_{\frac{11}{2},3}\left(\frac{1}{3}\mathfrak{c}^{\frac{1}{2}}\right)$*, thus*

$$|y_1(\mathfrak{c}) - y_0(\mathfrak{c})| < \frac{1}{2}\mathfrak{c}^2\mathbb{E}_{\frac{11}{2},3}\left(\frac{1}{3}\mathfrak{c}^{\frac{1}{2}}\right).$$

Using MATLAB, the solution of (5.2) *is computed and depicted in Figure 2. An error of the approximate solution* $y_1(\mathfrak{c})$ *can be estimated.*

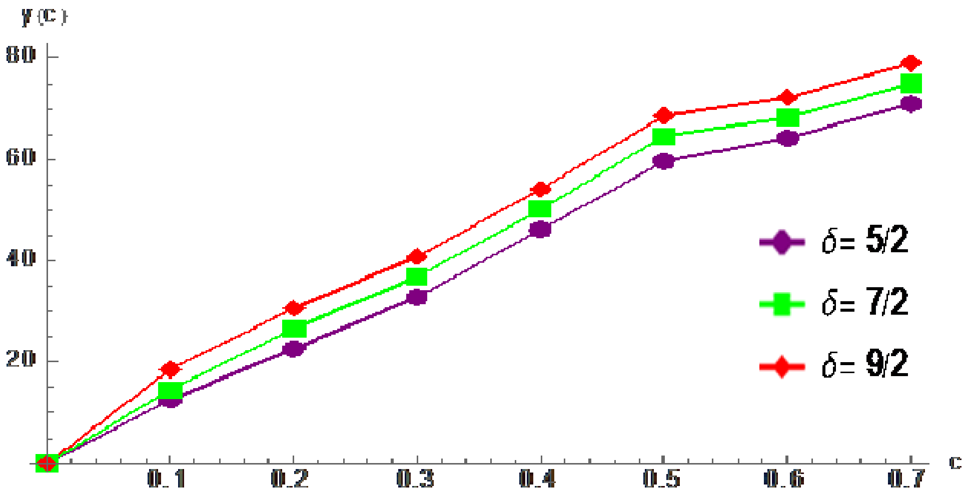

Figure 2. Solution of Eq (5.2).

6. Conclusions

This paper may be divided into three main parts, the framework of Hyers-Ulam Mittag Leffler problem of fractional derivatives and nonlinear fractional integrals, namely two Caputo fractional derivatives using a fractional Fourier transform. the fractional calculus, the theory of linear fractional differential equations and examples of the fractional calculus. In the beginning, we recalled some techniques, classes of functions and basic integral transforms which are necessary for further investigation of the fractional calculus rules. Then we introduced some standard approaches to the definition of fractional differential equations, namely the Riemann-Liouville and the two Caputo fractional approaches and the sequential fractional derivative, and studied their basic properties. In particular, we formulate the theorem describing the structure of the Hyers-Ulam Mittag Leffler problem for linear two-term equations. In particular, we derive the two Caputo fractional derivative step response functions of those generalized systems. We gave some examples of the fractional differential equation for important functions like the power function and functions of the Mittag-Leffler type. Finally, we considered the fractional differential equation of a discontinuous function where some of their effects were demonstrated.

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Conflict of interest

The authors declare that they have no competing interests.

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