

# THE FIRST INTEGRAL METHOD FOR THE (3+1)-DIMENSIONAL MODIFIED KORTEWEG-DE VRIES-ZAKHAROV-KUZNETSOV AND HIROTA EQUATIONS

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The first integral method is applied to get the different types of solutions of the (3+1)-dimensional modified Korteweg-de Vries-Zakharov-Kuznetsov and Hirota equations. We obtain envelope, bell shaped, trigonometric, and kink soliton solutions of these nonlinear evolution equations. The applied method is an effective one to obtain different types of solutions of nonlinear partial differential equations.

*Key words*: First integral method, Modified Korteweg-de Vries equation, Zakharov-Kuznetsov equation, Hirota equation, Analytical solutions.

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## 1. INTRODUCTION

The mathematical modelling of nonlinear physical systems is generally expressed in terms of nonlinear evolution equations. Therefore, it is crucial to obtain the most general solutions of the corresponding nonlinear partial differential equations (NPDEs) describing the evolution of such nonlinear systems. The general solutions of the NPDEs provide a lot of information about the intrinsic structure of such equations. Many effective numerical and analytical methods have been used to solve such NPDEs in different areas of nonlinear science, such as nonlinear hydrodynamics, nonlinear optics and photonics, Bose-Einstein condensate, nonlinear plasma physics, etc.; see, for example, Refs. [1]-[15] and references cited therein.

We recall that most of these methods use the wave variable transformation to reduce the NPDEs to ordinary differential equations (ODEs) in order to acquire the corresponding solutions. Several powerful techniques have been used, such as, Tanh [16],  $G'/G$ -expansion [17], Jacobi elliptic function [18], inverse scattering [19],

Hirota bilinear [20], exp-function [21], and first integral method [22]. All of these techniques are effective methods for acquiring travelling wave solutions of NPDEs. The first integral method (FIM) has been used by Feng [22] to solve the Burgers-Korteweg-de Vries equation. This method has been successfully implemented to other NPDEs and to some fractional differential equations which are new types of differential equations. For more details on this topic we refer the readers to Refs. [23]-[25]. In recent years, many studies on this method have been reported. Raslan [26] has used this method for the Fisher equation. Tascan and Bekir [27] have used the same method for Cahn-Allen equation, and Abbasbandy and Shirzadi [28] have investigated Benjamin-Bona-Mohany equation by using FIM. Also, Jafari *et al.* [29] have recently investigated the Biswas-Milovic equation, and so on [30]-[37].

In this paper we present the method in detail in Section 2. FIM has been applied to the modified Korteweg-de Vries-Zakharov-Kuznetsov (mKdVZK) equation [36] and Hirota equation [37] in Section 3. We give some conclusions in the last section.

## 2. THE FIRST INTEGRAL METHOD

The principal structures of the FIM are as follows:

**Step 1.** We consider a typical NPDE:

$$W(q, q_t, q_x, q_{xt}, q_{tt}, q_{xx}, \dots) = 0 \quad (1)$$

then the Eq. (1) transforms to an ODE as

$$H(Q, Q', Q'', Q''', \dots) = 0 \quad (2)$$

such that  $\xi = x \mp ct$  and  $Q' = \partial Q(\xi)/\partial \xi$ .

**Step 2.** It could be taken an ODE (2) as:

$$q(x, t) = q(\xi). \quad (3)$$

**Step 3.** A new independent variable is given by

$$Q(\xi) = q(\xi), G(\xi) = \partial q(\xi)/\partial \xi \quad (4)$$

that gives a new system of ODEs

$$\begin{aligned} \partial Q(\xi)/\partial \xi &= G(\xi) \\ \partial F(\xi)/\partial \xi &= P(Q(\xi), G(\xi)) \end{aligned} \quad (5)$$

**Step 4.** In accordance with the qualitative theory of ODEs [38], if it is possible to get the first integrals for the system (5), it could be immediately obtained the solutions of the system (5). The division theorem (DT) [39] gives us an idea how to obtain such first integrals.

### 3. APPLICATIONS

In this section we illustrate the FIM for both the mKdVZK and Hirota equations.

#### 3.1. THE mKdVZK EQUATION

We illustrate the FIM for the (3+1)-dimensional mKdVZK equation [33]

$$q_t + \beta q^2 q_x + q_{xxx} + q_{xyy} + q_{xzz} = 0. \quad (6)$$

Equation (6) turns to the following ODE by using the wave variable  $\xi = x + y + z - \lambda t$ :

$$-\lambda Q_\xi + \frac{1}{3}\beta Q^2 Q_\xi + 3Q_{\xi\xi\xi} = 0. \quad (7)$$

Each side of the Eq. (7) is integrated once

$$-\lambda Q + \frac{1}{3}\beta Q^3 + 3Q_{\xi\xi} = n, \quad (8)$$

where  $n$  is a constant of integration.

Then, using (3) and (4) we obtain

$$\begin{aligned} Q'(\xi) &= G(\xi) \\ G'(\xi) &= \frac{n}{3} + \frac{\lambda}{3}Q(\xi) - \frac{\beta}{9}Q^3(\xi) \end{aligned} \quad (9)$$

In accordance with the FIM, it is supposed that  $Q(\xi)$  and  $G(\xi)$  are non-trivial solutions of system (9) and  $F(Q, G) = \sum_{i=0}^r a_i(Q)G^i$  is an irreducible function in the domain  $C[Q, G]$  such that

$$F(Q(\xi), G(\xi)) = \sum_{i=0}^r a_i(Q)G^i = 0 \quad (10)$$

where  $a_i(Q), (i = 0, 1, 2, \dots, m)$  are polynomials of  $Q$  and  $a_m(Q) \neq 0$ . Eq. (10) is a first integral for the system (10), owing to the DT, there exists  $g(Q) + h(Q)G$  in  $C[Q, G]$  such as:

$$\begin{aligned} dF/d\xi &= \frac{dF}{dQ} \frac{dQ}{d\xi} + \frac{dF}{dG} \frac{dG}{d\xi} \\ &= [g(Q) + h(Q)G] \sum_{i=0}^r a_i(Q)G^i \end{aligned} \quad (11)$$

In this study, we consider  $r = 1$  and  $r = 2$  in Eq. (11).

**Case 1.**

If we equate the coefficients of  $G^i$  ( $i = 0, 1, 2, \dots, r$ ) of Eq. (11) for  $r = 1$ , we have

$$\dot{a}_1(Q) = h(Q)a_1(Q) \quad (12)$$

$$\dot{a}_0(Q) = a_1(Q)g(Q) + h(Q)a_0(Q) \quad (13)$$

$$a_0(Q)g(Q) = a_1(Q)\left(\frac{n}{3} + \frac{\lambda}{3}Q(\xi) - \frac{\beta}{9}Q^3(\xi)\right) \quad (14)$$

Since  $a_1(Q)$  ( $i = 0, 1$ ) is a polynomial of  $Q$ ,  $a_1(Q)$  is a constant and  $h(Q) = 0$  from (12). For convenience, it is obtained  $a_1(Q) = 1$ , and by equalization of the degrees of  $g(Q)$  and  $a_0(Q)$  we conclude that the degree of  $g(Q)$  is equal to zero. Then, we assume that  $g(Q) = A_0Q + A_1$ , and we obtain from Eq. (13)

$$a_0(Q) = \frac{A_0}{2}Q^2 + A_1Q + A_2 \quad (15)$$

Replacing  $a_0(Q)$ ,  $a_1(Q)$ , and  $g(Q)$  in Eq. (14), and equating the coefficients of  $Q^i$  to zero, we have:

$$A_0 = \frac{i}{3}\sqrt{2\beta}, A_1 = 0, A_2 = \frac{\lambda}{i\sqrt{2\beta}} \quad (16)$$

$$A_0 = -\frac{i}{3}\sqrt{2\beta}, A_1 = 0, A_2 = -\frac{\lambda}{i\sqrt{2\beta}}, i^2 = -1 \quad (17)$$

Setting (16) and (17) in (10), we have

$$Q'(\xi) = \frac{i}{6}\sqrt{2\beta}Q^2(\xi) + \frac{\lambda}{i\sqrt{2\beta}} \quad (18)$$

$$Q'(\xi) = -\frac{i}{6}\sqrt{2\beta}Q^2(\xi) - \frac{\lambda}{i\sqrt{2\beta}} \quad (19)$$

If we solve the Eqs. (18) and (19) by using (3) and (4) respectively, we get the following analytical solutions of Eq. (6):

$$q(x, y, z, t) = i\sqrt{3\lambda/\beta}\tan[1/6(\mp\sqrt{6\lambda}(x + y + z - \lambda t) + 6i\sqrt{3\beta\lambda}c_0)], \quad (20)$$

where  $c_0$  is an auxiliary constant.

In Fig. 1 we plot the typical kink solution (20) to the mKdVZK equation for the parameters:  $\lambda = c_0 = 1$ ,  $\beta = 3$ , and  $y = z = 1$ .

**Case 2.**

If we equate the coefficients of  $G^i$  ( $i = 0, 1, 2, \dots, r$ ) of Eq. (11) for  $r = 2$ , we have

$$\dot{a}_2(Q) = h(Q)a_2(Q) \quad (21)$$

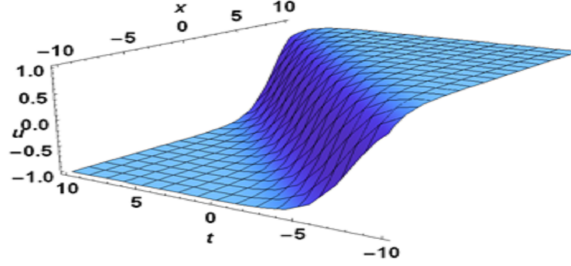


Fig. 1 – (Color online). The figure displays kink solution (20) to the mKdVZK equation for  $\lambda = c_0 = 1, \beta = 3, y = z = 1$ .

$$\dot{a}_1(Q) = a_2(Q)g(Q) + h(Q)a_1(Q) \quad (22)$$

$$a_1(Q)g(Q) + h(Q)a_0(Q) = \dot{a}_0(Q) + 2a_2(Q)\left(\frac{n}{3} + \frac{\lambda}{3}Q - \frac{\beta}{9}Q^3\right) \quad (23)$$

$$a_1(Q)\dot{G} = a_0(Q)g(Q) \quad (24)$$

Since  $a_2(Q)$  ( $i = 0, 1, 2$ ) is a polynomial of  $Q$ , we conclude that  $a_2(Q)$  is a constant and  $h(Q) = 0$  from (12). For convenience, it is obtained  $a_2(Q) = 1$ , and by equalization the degrees of  $g(Q)$ ,  $a_1(Q)$ , and  $a_2(Q)$  we conclude that the degree of  $g(Q)$  is equal to one. Then, we suppose that  $g(Q) = A_0Q + A_1$ , and we obtain from Eq. (22) the result

$$a_1(Q) = \frac{A_0}{2}Q^2 + A_1Q + A_2 \quad (25)$$

Replacing  $a_0(Q)$ ,  $a_1(Q)$ ,  $a_2(Q)$  and  $g(Q)$  in Eq. (23) and equating the coefficients of  $Q^i$  to zero, we have

$$A_0 = \frac{2}{3i}\sqrt{2\beta}, A_1 = 0, A_2 = \frac{4\lambda}{2i\sqrt{2\beta}} \quad (26)$$

$$A_0 = -\frac{2}{3i}\sqrt{2\beta}, A_1 = 0, A_2 = -\frac{4\lambda}{2i\sqrt{2\beta}} \quad (27)$$

By setting (26) and (27) in (10), we have

$$Q'(\xi) = \frac{2}{3i}\sqrt{2\beta}Q^2(\xi) + \frac{4\lambda}{2i\sqrt{2\beta}} \quad (28)$$

$$Q'(\xi) = -\frac{2}{3i}\sqrt{2\beta}Q^2(\xi) - \frac{4\lambda}{2i\sqrt{2\beta}} \quad (29)$$

If we solve the Eqs. (28) and (29) by using (3) and (4), respectively, we get the analytical solutions of Eq. (6):

$$q(x, y, z, t) = i\sqrt{3\lambda/\beta}\tan\left[-\frac{\sqrt{6\lambda}}{3}(x + y + z - \lambda t)\right] \quad (30)$$

$$q(x, y, z, t) = i\sqrt{3\lambda/\beta}\tan\left[\frac{\sqrt{6\lambda}}{3}(x + y + z - \lambda t)\right], \quad (31)$$

where (20), (30), and (31) are trigonometric soliton solutions for Eq. (6).

Now, we deduce that  $\deg[g(Q)] = 0$ , only. Then, we suppose that  $g(Q) = A$ , and we consider  $a_2(Q) = 1$ . Equating the degrees of  $g(Q), a_1(Q), a_2(Q)$  we obtain from Eq. (22) the result

$$a_0(Q) = A_4Q^4 + A_3Q^3 + A_2Q^2 + A_1Q + A_0 \quad (32)$$

Replacing  $a_0(Q), a_1(Q), a_2(Q)$  and  $g(Q)$  in Eq. (23) and equating the coefficients of  $Q^i$  to zero, we have

$$A_0 = A_0, A_1 = 0, A_2 = \frac{-\lambda}{3}, A_3 = 0, A_4 = \frac{\beta}{18}, \beta \neq 0 \quad (33)$$

By setting (33) in (10), we have

$$Q'(\xi) = \sqrt{\frac{\beta}{18}Q^4(\xi) - \frac{\lambda}{3}Q^2(\xi) + A_0} \quad (34)$$

If we solve the Eq. (34) by using (3) and (4), we have:

i) For  $A_0 = 0, \lambda < 0, \beta < 0$  we get a bell-shaped solitary wave solution of Eq. (6)

$$q(x, y, z, t) = \sqrt{6\lambda/\beta}\operatorname{sech}\left[\sqrt{\lambda/3}(x + y + z - \lambda t)\right] \quad (35)$$

ii) For  $A_0 = 0, \lambda > 0, \beta > 0$ , we get a trigonometric solution:

$$q(x, y, z, t) = \sqrt{6\lambda/\beta}\operatorname{sec}\left[i\sqrt{\lambda/3}(x + y + z - \lambda t)\right] \quad (36)$$

iii) For  $A_0 = \frac{\lambda^2}{2\beta}, \lambda > 0, \beta > 0$  Eq. (6) has a kink-shaped solitary wave solution:

$$q(x, y, z, t) = \pm\sqrt{3\lambda/\beta}\operatorname{tanh}\left[\pm i\sqrt{\lambda/6}(x + y + z - \lambda t)\right] \quad (37)$$

iv) For  $A_0 = \frac{\lambda^2}{2\beta}, \lambda < 0, \beta > 0$ , a trigonometric soliton solution is obtained:

$$q(x, y, z, t) = -i\sqrt{3\lambda/\beta}\tan\left[-\sqrt{\lambda/6}(x + y + z - \lambda t)\right] \quad (38)$$

v) If  $A_1 = 0, A_3 = 0$ , Eq. (34) has two cnoidal-type solutions

$$Q(\xi) = \sqrt{\frac{6\lambda m^2}{\beta(2m^2 - 1)}}\operatorname{cn}\left[\frac{-\lambda}{3(2m^2 - 1)}\xi\right], \quad (39)$$

$$\beta < 0, \lambda < 0, A_0 = \frac{(m^2 - 1)\lambda}{2m^2 - 1}$$

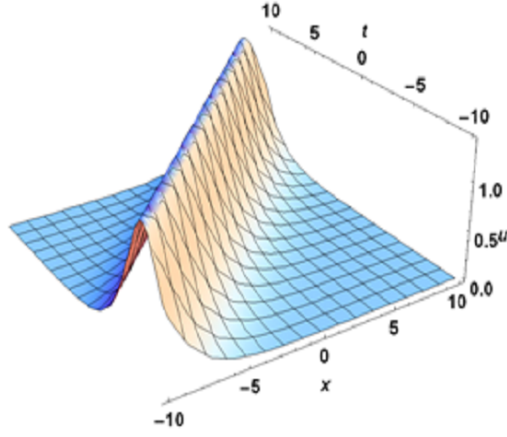


Fig. 2 – (Color online). The figure displays bell-shaped soliton solution (35) to the mKdVZK equation  $\lambda = 1, \beta = 3, y = z = 1$ .

and

$$Q(\xi) = \sqrt{\frac{6\lambda m^2}{\beta(m^2+1)}} \operatorname{sn}\left[\frac{\lambda}{3(m^2+1)}\xi\right], \quad (40)$$

$$\beta > 0, \lambda > 0, A_0 = \frac{(\lambda m)^2}{9(m^2+1)}$$

As  $m \rightarrow 1$ , the snoidal-type solutions (39) and (40) degenerate to the solutions (35) and (36), respectively

### 3.2. HIROTA EQUATION

We consider the Hirota equation [34]

$$iu_t + u_{xx} + 2|u|^2u + i\alpha u_{xxx} + 6i\alpha|u|^2u_x = 0, \quad (41)$$

where  $i^2 = -1$ .

We make the transformation  $u = e^{iU}Q(\xi)$ ,  $U = px + qt$  and  $\xi = kx + ct$ . Substituting these transformations by separating real and imaginary parts into Eq. (41) yields

$$(c + 2pk - 3\alpha p^2k)Q_\xi + 6\alpha kQ^2Q_\xi + \alpha k^3Q_{\xi\xi\xi} = 0 \quad (42)$$

$$(\alpha p^3 - q - p^2)Q + 2(1 - 3\alpha p)Q^3 + (1 - 3\alpha p)k^2Q_{\xi\xi} = 0 \quad (43)$$

By integrating Eq. (42), we have

$$(c + 2pk - 3\alpha p^2k)Q + 2\alpha kQ^3 + \alpha k^3Q_{\xi\xi} = n, \quad (44)$$

where  $n$  is an integration constant.

By equating the coefficients of the same terms in Eq. (43) and Eq. (44), we have

$$\begin{aligned}\alpha p^3 - q - p^2 &= c + 2pk - 3\alpha p^2 k, \\ 2(1 - 3\alpha p) &= 2\alpha k, \\ (1 - 3\alpha p)k^2 &= \alpha k^3.\end{aligned}\tag{45}$$

From (45) it is obtained that  $p = (1 - \alpha k)/3\alpha$  and  $q = \frac{-2 - 27a^2c - 6ak + 8a^3k^3}{27a^2}$ . Then, using (3) and (4) we have

$$\begin{aligned}\dot{Q}(\xi) &= G(\xi) \\ \dot{G}(\xi) &= \frac{3\alpha p^2 k - c - 2pk}{\alpha k^3} Q - \frac{2}{k^2} Q^3\end{aligned}\tag{46}$$

In accordance with the FIM, it is supposed that  $Q(\xi)$  and  $G(\xi)$  are non-trivial solutions of system (46) and  $F(Q, G) = \sum_{i=0}^r a_i(Q)G^i$  is an irreducible function in the domain  $C[Q, G]$  such that in Eqs. (10) and (11). We get:

**Case 3.**

If we equate the coefficients of  $G^i (i = 0, 1, 2, \dots, r)$  on both sides of Eq. (11) for  $r = 1$ , we have

$$\dot{a}_1(Q) = h(Q)a_1(Q)\tag{47}$$

$$\dot{a}_0(Q) = a_1(Q)g(Q) + h(Q)a_0(Q)\tag{48}$$

$$a_0(Q)g(Q) = a_1(Q)\left(\frac{3\alpha p^2 k - c - 2pk}{\alpha k^3} Q - \frac{2}{k^2} Q^3\right)\tag{49}$$

Since  $a_1(Q) (i = 0, 1)$  is polynomial of  $Q$ ,  $a_1(Q)$  is a constant and  $h(Q) = 0$  from (47). For convenience, it is obtained  $a_1(Q) = 1$ , and equalization the degrees of  $g(Q)$  and  $a_0(Q)$  we conclude the degree of  $g(Q)$  is equal to zero. Then, we assume that  $g(Q) = A_0Q + A_1$ , and we obtain from Eq. (48) as follows

$$a_0(Q) = \frac{A_0}{2} Q^2 + A_1 Q + A_2\tag{50}$$

Replacing  $a_0(Q), a_1(Q)$  and  $g(Q)$  in Eq. (14) then equating the coefficients of  $Q^i$  to zero, we have:

$$A_0 = \mp \frac{2i}{k}, \quad A_1 = 0, \quad A_2 = \mp \frac{i}{2k^2\alpha} (-c - 2pk + 3\alpha p^2 k)\tag{51}$$

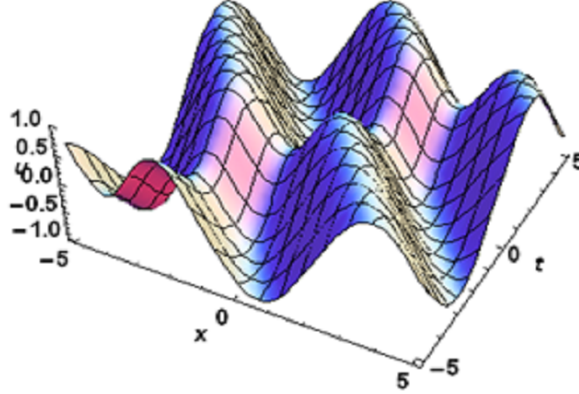


Fig. 3 – (Color online). The figure depicts the envelope solution (52) to the Hirota equation; here  $\alpha = 1/4, k = c = -3/4, p = k = 1, q = 3/4, c_1 = 0$ .

Substituting (51) in (50), we have analytical solutions of Eq. (41)

$$Q(\xi) = i\sqrt{\frac{c + kp(2 - 3p\alpha)}{2k\alpha}} \tanh \left[ \frac{c + kp(2 - 3p\alpha)}{2k\alpha} (\xi - 2ik^2c_1) \right] \quad (52)$$

$$u(x, t) = i\sqrt{\frac{c + kp(2 - 3p\alpha)}{2k\alpha}} \tanh \left[ \frac{c + kp(2 - 3p\alpha)}{2k\alpha} (kx + ct - 2ik^2c_1) \right] e^{i(px+qt)} \quad (53)$$

where  $c_1$  is an auxiliary constant and the (52) is envelope kink soliton solution of (41).

**Case 4.**

If we equate the coefficients of  $G^i (i = 0, 1, 2, \dots, r)$  of Eq. (11) for  $r = 2$ , we have

$$\dot{a}_2(Q) = h(Q)a_2(Q) \quad (54)$$

$$\dot{a}_1(Q) = a_2(Q)g(Q) + h(Q)a_1(Q) \quad (55)$$

$$a_1(Q)g(Q) + h(Q)a_0(Q) = \dot{a}_0(Q) + 2a_2(Q)\left(\frac{3\alpha p^2 k - c - 2pk}{\alpha k^3}Q - \frac{2}{k^2}Q^3\right) \quad (56)$$

$$a_1(Q)\dot{G} = a_0(Q)g(Q) \quad (57)$$

Since  $a_2(Q) (i = 0, 1, 2)$  is polynomial of  $Q$ , we conclude that  $a_2(Q)$  is a constant and  $h(Q) = 0$  from (54). For convenience, it is obtained  $a_2(Q) = 1$ , and from

equalization of the degrees of  $g(Q)$ ,  $a_1(Q)$ , and  $a_2(Q)$  we conclude that the degree of  $g(Q)$  is equal to one. Then, we suppose that  $g(Q) = A_0Q + A_1$ , and we obtain from Eq. (56) as follows

$$a_1(Q) = \frac{A_0}{2}Q^2 + A_1Q + A_2 \quad (58)$$

$$a_0 = \left(8 + \frac{A_0^2 k^2}{8k^2}\right)Q^4 + \frac{1}{2}A_0A_1Q^3 + \left(\frac{A_0A_2}{2} + \frac{4i\sqrt{d} + A_1^2 k}{2k}\right)Q^2 + A_1A_2Q + d, \quad (59)$$

where  $d$  is a constant.

Replacing  $a_0(Q)$ ,  $a_1(Q)$ ,  $a_2(Q)$ , and  $g(Q)$  in Eq. (57) and equating the coefficients of  $Q^i$  to zero, we have

$$c = \mp i(2ikp + 2\sqrt{d}k^2\alpha - 3ikp^2\alpha), \quad A_1 = 0, \quad (60)$$

$$A_2 = \mp 2\sqrt{d}, \quad A_0 = \frac{2(-c - 2pk + 3\alpha p^2 k)}{\sqrt{d}k^3\alpha}$$

Setting (60) in (10), we have

$$Q' = \frac{(-c - 2pk + 3\alpha p^2 k)}{\sqrt{d}k^3\alpha}Q^2 \mp 2\sqrt{d} \quad (61)$$

$$Q' = \sqrt{d - \frac{2i\sqrt{d}}{k}Q^2 - \frac{1}{k^2}Q^4} \quad (62)$$

If we solve the Eq. (61) by using (3) and (4), we have

$$Q(\xi) = -\sqrt[4]{-dk^2} \tan \left[ \sqrt[4]{\frac{-d}{k^2}}(2\xi + kc_2) \right] \quad (63)$$

$$u(x, t) = -\sqrt[4]{-dk^2} \tan \left[ \sqrt[4]{\frac{-d}{k^2}}(2(kx + ct) + kc_2) \right] e^{i(px+qt)},$$

where  $c_2$  is an arbitrary constant and the equation (63) is the envelope trigonometric soliton solution of (41).

If we solve the Eq. (62), we get

i)

$$Q(\xi) = \sqrt{q} \tan \left[ \sqrt{-\frac{i\sqrt{d}}{k}} \xi \right], -1/k^2 > 0, -\frac{2i\sqrt{d}}{k} > 0, q = ik\sqrt{d},$$

$$u(x, t) = \sqrt{q} \tan \left[ \sqrt{-\frac{i\sqrt{d}}{k}} (kx + ct) \right] e^{i(px+qt)} \quad (64)$$

ii)

$$Q(\xi) = \sqrt{-q} \tanh \left[ \sqrt{-\frac{i\sqrt{d}}{k}} \xi \right], -1/k^2 > 0, -\frac{2i\sqrt{d}}{k} < 0, q = ik\sqrt{d},$$

$$u(x, t) = \sqrt{-q} \tanh \left[ \sqrt{-\frac{i\sqrt{d}}{k}} (kx + ct) \right] e^{i(px+qt)} \quad (65)$$

iii)

$$Q(\xi) = \sqrt{\frac{2qm^2}{1-2m^2}} \operatorname{cn} \left[ \frac{-2q}{k^2(1-2m^2)} \xi \right], -1/k^2 < 0, -\frac{2i\sqrt{d}}{k} > 0, d \neq 0,$$

$$u(x, t) = \sqrt{\frac{2qm^2}{1-2m^2}} \operatorname{cn} \left[ \frac{-2q}{k^2(1-2m^2)} (kx + ct) \right] e^{i(px+qt)}. \quad (66)$$

iv)

$$Q(\xi) = \sqrt{\frac{2qm^2}{1+m^2}} \operatorname{sn} \left[ \frac{2q}{k^2(1+m^2)} \xi \right], -1/k^2 > 0, -\frac{2i\sqrt{d}}{k} < 0, d \neq 0,$$

$$u(x, t) = \sqrt{\frac{2qm^2}{1+m^2}} \operatorname{sn} \left[ \frac{2q}{k^2(1+m^2)} (kx + ct) \right] e^{i(px+qt)}. \quad (67)$$

v)

$$Q(\xi) = \sqrt{\frac{2qm^2}{m^2-2}} \operatorname{dn} \left[ \frac{2q}{k^2(m^2-2)} \xi \right], -1/k^2 < 0, -\frac{2i\sqrt{d}}{k} > 0, d \neq 0,$$

$$u(x, t) = \sqrt{\frac{2qm^2}{m^2-2}} \operatorname{dn} \left[ \frac{2q}{k^2(m^2-2)} (kx + ct) \right] e^{i(px+qt)}. \quad (68)$$

The solution (64) is the envelope trigonometric soliton, the solution (65) is the envelope kink soliton, and the solutions (66)-(68) are the corresponding snoidal and cnoidal solitons.

Now, we deduce that  $\deg[g(Q)] = 0$ , only. Then, we suppose that  $g(Q) = A$ , and we consider  $a_2(Q) = 1$ . Equating the degrees of  $g(Q)$ ,  $a_1(Q)$ , and  $a_2(Q)$  we

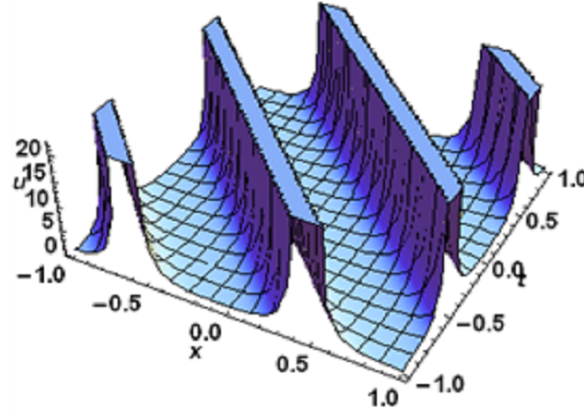


Fig. 4 – (Color online). The figure shows the envelope periodic wave solution (66) to the Hirota equation; here  $k = -2, \alpha = p = 1, a = c = 2, q = -2.8$ .

obtain from Eq. (57) as follows

$$a_0(X) = A_4Q^4 + A_3Q^3 + A_2Q^2 + A_1Q + A_0 \quad (69)$$

Replacing  $a_0(Q), a_1(Q), a_2(Q)$ , and  $g(Q)$  in Eq. (56) and setting the coefficients  $Q^i$  to zero, we have

$$A_1 = 0, A_2 = -\frac{3p^2}{k^2} + \frac{c}{k^3\alpha} + \frac{p}{k^2\alpha}, A_3 = 0, A_4 = \frac{1}{k^2} \quad (70)$$

By setting (33) in (10), we get a differential equation as follows

$$Q'(\xi) = \sqrt{\frac{1}{k^2}Q^4(\xi) + \left(-\frac{3p^2}{k^2} + \frac{c}{k^3\alpha} + \frac{p}{k^2\alpha}\right)Q^2(\xi) + A_0} \quad (71)$$

If we solve the above differential equation, we obtain

$$\begin{aligned} Q(\xi) &= \frac{c + kp(1 - 3p\alpha)}{2k^3\alpha} \tan \left[ \frac{c + kp(1 - 3p\alpha)}{2k^3\alpha} \xi \right], \\ A_0 &= \frac{(c + kp(1 - 3p\alpha))^2}{4k^4\alpha^2}, -\frac{3p^2}{k^2} + \frac{c}{k^3\alpha} + \frac{p}{k^2\alpha} > 0, \\ u(x, t) &= \frac{c + kp(1 - 3p\alpha)}{2k^3\alpha} \tan \left[ \frac{c + kp(1 - 3p\alpha)}{2k^3\alpha} (kx + ct) \right] e^{i(px+qt)}, \end{aligned} \quad (72)$$

$$\begin{aligned}
Q(\xi) &= -\frac{c+kp(1-3p\alpha)}{2k^3\alpha} \tanh \left[ \sqrt{-\frac{c+kp(1-3p\alpha)}{2k^3\alpha}} \xi \right], \\
A_0 &= \frac{(c+kp(1-3p\alpha))^2}{4k^4\alpha^2}, -\frac{3p^2}{k^2} + \frac{c}{k^3\alpha} + \frac{p}{k^2\alpha} < 0, \\
u(x,t) &= \frac{c+kp(1-3p\alpha)}{2k^3\alpha} \tanh \left[ -\frac{c+kp(1-3p\alpha)}{2k^3\alpha} (kx+ct) \right] e^{i(px+qt)},
\end{aligned} \tag{73}$$

$$\begin{aligned}
Q(\xi) &= \sqrt{-\frac{m^2(c+kp(1-3p\alpha))}{k(1+m^2)\alpha}} \operatorname{sn} \left[ \sqrt{-\frac{c+kp(1-3p\alpha)}{k^3(1+m^2)\alpha}} \xi \right], \\
A_0 &= \frac{m^2(c+kp(1-3p\alpha))^2}{k^4(1+m^2)\alpha^2}, -\frac{3p^2}{k^2} + \frac{c}{k^3\alpha} + \frac{p}{k^2\alpha} < 0, \\
u(x,t) &= \sqrt{-\frac{m^2(c+kp(1-3p\alpha))}{k^4(1+m^2)\alpha^2}} \\
&\quad \cdot \operatorname{sn} \left[ \sqrt{-\frac{c+kp(1-3p\alpha)}{k^3(1+m^2)\alpha}} (kx+ct) \right] e^{i(px+qt)}.
\end{aligned} \tag{74}$$

The solution (72) is the envelope trigonometric soliton, the solution (73) is the envelope kink soliton, and the solution (74) is the snoidal wave. As  $m \rightarrow 1$ , the snoidal solution (74) degenerates to the solution (73).

#### 4. CONCLUSION

We have used the first integral method to obtain several new exact solutions of the modified Korteweg-de Vries-Zakharov-Kuznetsov and Hirota equations. We have acquired different types of exact solutions that are denoted in terms of trigonometric, trigonometric, algebraic, cnoidal, snoidal, and exponential functions. Some of the obtained solutions are new, to the best of our knowledge. Consequently, the first integral method is extremely effective to construct different types of exact soliton solutions of both nonlinear partial differential equations and coupled systems of such nonlinear evolution equations.

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