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Common fixed point theorems for generalized (ϕ, ψ) -weak contraction condition in complete metric spaces

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Abstract

The intent of this manuscript is to establish some common fixed point theorems in a complete metric space under weak contraction condition for two pairs of discontinuous weak compatible maps. The results proved herein are the generalization of some recent results in literature. We give an example to support our results.

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1 Introduction

Let (X, d) be a metric space. A function $T : (X, d) \rightarrow (X, d)$ is said to satisfy the contraction condition if there exists a real number $k \in (0, 1)$ and for all $x, y \in X$ such that

$$d(Tx, Ty) \leq kd(x, y). \quad (1.1)$$

It is remarkable that a self-map T satisfying condition (1.1) implies that T is continuous (see [1]). One can see in the literature on metric fixed point theory that condition (1.1) has been extended and generalized by fixed point theorists in many ways for obtaining fixed points, common fixed points and, very recently, proximal points in different spaces.

The title of this paper implies that the pair of maps in a complete metric space satisfies certain inequality by generalizing the contraction condition of Banach, but the inequality itself does not force the mapping to be continuous. It was an open question for almost four decades after Banach's fixed point theorem [1]. In 1968, Kannan [2] answered this question affirmatively by introducing the following inequality:

$$d(Tx, Ty) \leq \beta [d(x, Tx) + d(y, Ty)] \quad \text{for all } x, y \in X \text{ and } \beta \in \left(0, \frac{1}{2}\right). \quad (1.2)$$

Rakotch [3] and later Boyd and Wond [4] generalized the inequality of Banach by introducing a *control function* as follows:

$$d(Tx, Ty) \leq \alpha(x, y)d(x, y) \quad \text{for all } x, y \in X \text{ and } \alpha : [0, \infty) \rightarrow [0, 1) \quad (1.3)$$

and

$$d(Tx, Ty) \leq \phi(d(x, y)) \quad \text{for all } x, y \in X \text{ and } \phi : [0, \infty) \rightarrow [0, \infty), \tag{1.4}$$

and it satisfies certain conditions. In a Hilbert space, Alber and Guerre-Delabriere [5] introduced a weak contraction condition, which was later extended by Rhoades [6] in a complete metric space, and they argued that the result of Alber and Guerre-Delabriere [5] still holds in a complete metric space.

A mapping $T : X \rightarrow X$ satisfies the following condition:

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) \tag{1.5}$$

for all $x, y \in X$, and $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$.

Remark If $\varphi(t) = (1 - k)t$, where $k \in (0, 1)$ in (1.5), then we shall have the condition (1.1) of Banach [1].

So, in view of (1.1), condition (1.5) is a weaker condition, and we call this condition a weak contraction condition.

Definition 1.1 (Altering distance function [7]) A function $\psi : [0, \infty) \rightarrow [0, \infty)$ is called an altering distance function if it satisfies the following conditions:

- (1) ψ is monotone increasing and continuous,
- (2) $\psi(t) = 0$ if and only if $t = 0$.

Definition 1.2 ([8]) A pair of self-mappings A and B of a metric space (X, d) is said to be weakly compatible if they commute at their coincidence points. In other words, if $Ax = Bx$ for some $x \in X$, then $ABx = BAx$.

The first and second author of this paper with Choudhury [9] proved a common fixed point theorem in Saks spaces. The main purpose of this note is to establish a few common fixed point theorems by generalizing the results of Murthy *et al.* [9] for two pairs of discontinuous functions in a complete metric space by using a weaker condition than condition (1.5) called (φ, ψ) -weak contraction condition in metric spaces. For example, refer to Rhoades [6], Dutta and Choudhury [10], Zhang and Song [11], Doric [12], Hosseini [13], Abkar and Choudhury [14], Ahmad *et al.* [15], Hussain *et al.* [16], *etc.*

2 Fixed point theorems in a complete metric space

Theorem 2.1 Let (X, d) be a complete metric space, and let A, B, S and $T : X \rightarrow X$ be mappings satisfying

$$\psi(d(Ax, By)) \leq \psi(M(x, y)) - \phi(N(x, y)) \tag{2.1}$$

for all $x, y \in X$, with $x \neq y$ and

$$M(x, y) = \max \left\{ d(Sx, Ty), \frac{1}{2}(d(Sx, Ax) + d(Ty, By)), \frac{1}{2}(d(Sx, By) + d(Ty, Ax)) \right\}$$

and

$$N(x, y) = \min \left\{ d(Sx, Ty), \frac{1}{2} (d(Sx, Ax) + d(Ty, By)), \frac{1}{2} (d(Sx, By) + d(Ty, Ax)) \right\},$$

$$A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X), \tag{2.2}$$

$$(A, S) \quad \text{and} \quad (B, T) \quad \text{are weak compatible pairs}, \tag{2.3}$$

$$\phi : [0, \infty) \rightarrow [0, \infty) \quad \text{is such that } \phi(t) > 0, \tag{2.4}$$

which is lower semi-continuous for all $t > 0$ and ϕ is discontinuous at $t = 0$ with $\phi(0) = 0$,

$$\psi : [0, \infty) \rightarrow [0, \infty) \quad \text{is an altering distance function}. \tag{2.5}$$

Then A, B, S and T have a unique common fixed point in X .

Proof Let $x_0 \in X$ be an arbitrary point. Since $A(X) \subset T(X)$ and $B(X) \subset S(X)$, then there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$, and for $x_1 \in X$, there exists a point $x_2 \in X$ such that $Bx_1 = Sx_2$. Inductively, we can construct a sequence

$$\begin{aligned} y_{2n+1} &= Ax_{2n} = Tx_{2n+1}, \\ y_{2n+2} &= Bx_{2n+1} = Sx_{2n+2} \quad \text{for } n = 0, 1, 2, \dots \end{aligned}$$

We assume, for all $n \in N \cup \{0\}$,

$$y_{2n} \neq y_{2n+1}. \tag{2.6}$$

At first, we shall show that $d(y_{2n}, y_{2n+1}) \rightarrow 0$ as $n \rightarrow \infty$ for all $n \in N \cup \{0\}$, where N is a set of natural numbers.

For this, suppose that $x = x_{2n}, y = x_{2n+1}$ in (2.1), we have

$$\begin{aligned} \psi(d(Ax_{2n}, Bx_{2n+1})) &= \psi(d(y_{2n+1}, y_{2n+2})) \\ &\leq \psi(M(x_{2n}, x_{2n+1})) - \phi(N(x_{2n}, x_{2n+1})), \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max \left\{ d(y_{2n}, y_{2n+1}), \frac{1}{2} (d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})), \right. \\ &\quad \left. \frac{1}{2} (d(y_{2n}, y_{2n+2}) + d(y_{2n+1}, y_{2n+1})) \right\} \end{aligned}$$

and

$$\begin{aligned} N(x_{2n}, x_{2n+1}) &= \min \left\{ d(y_{2n}, y_{2n+1}), \frac{1}{2} (d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})), \right. \\ &\quad \left. \frac{1}{2} (d(y_{2n}, y_{2n+2}) + d(y_{2n+1}, y_{2n+1})) \right\}. \end{aligned}$$

Then, by the triangular inequality, we have

$$M(x_{2n}, x_{2n+1}) \leq \max \left\{ d(y_{2n}, y_{2n+1}), \frac{1}{2} (d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})), \frac{1}{2} (d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})) \right\}.$$

If

$$d(y_{2n}, y_{2n+1}) < d(y_{2n+1}, y_{2n+2}), \tag{2.8}$$

then we get

$$M(x_{2n}, x_{2n+1}) \leq d(y_{2n+1}, y_{2n+2}), \tag{2.9}$$

and (2.7) implies the following:

$$\psi(d(y_{2n+1}, y_{2n+2})) \leq \psi(M(x_{2n}, x_{2n+1})) - \phi(N(x_{2n}, x_{2n+1})) \leq \psi(M(x_{2n}, x_{2n+1})).$$

Using the monotonically increasing property of ψ function, we have

$$d(y_{2n+1}, y_{2n+2}) \leq M(x_{2n}, x_{2n+1}). \tag{2.10}$$

From (2.9) and (2.10), we have

$$M(x_{2n}, x_{2n+1}) = d(y_{2n+1}, y_{2n+2}). \tag{2.11}$$

Since

$$0 < |d(y_{2n+1}, y_{2n+2}) - d(y_{2n}, y_{2n+1})| \leq d(y_{2n}, y_{2n+2}), \tag{2.12}$$

we have $N(x_{2n}, x_{2n+1}) > 0$, then from (2.7), (2.11) and the property of ϕ and ψ functions, we have

$$\begin{aligned} \psi(d(y_{2n+1}, y_{2n+2})) &\leq \psi(d(y_{2n+1}, y_{2n+2})) - \phi(N(x_{2n}, x_{2n+1})) \\ &< \psi(d(y_{2n+1}, y_{2n+2})), \end{aligned}$$

which is a contradiction. Thus we have

$$d(y_{2n+1}, y_{2n+2}) \leq d(y_{2n}, y_{2n+1}). \tag{2.13}$$

So, we obtain the following:

$$M(x_{2n}, x_{2n+1}) = d(y_{2n}, y_{2n+1}), \tag{2.14}$$

$$N(x_{2n}, x_{2n+1}) = \frac{1}{2} (d(y_{2n}, y_{2n+2})). \tag{2.15}$$

Now putting (2.14) and (2.15) in (2.7), we have

$$\psi(d(y_{2n+1}, y_{2n+2})) \leq \psi(d(y_{2n}, y_{2n+1})) - \phi(N(x_{2n}, x_{2n+1})). \tag{2.16}$$

Therefore $d(y_{2n}, y_{2n+1})$ is a monotonically decreasing sequence of nonnegative real numbers, then there exists a number $r > 0$ such that

$$\lim_{n \rightarrow \infty} d(y_{2n}, y_{2n+1}) = r > 0. \tag{2.17}$$

By virtue of (2.6) and (2.12), we have $N(x_{2n}, x_{2n+1}) > 0$. Taking $n \rightarrow \infty$ in (2.16) and using (2.17), we get

$$\lim_{n \rightarrow \infty} \psi(d(y_{2n+1}, y_{2n+2})) \leq \lim_{n \rightarrow \infty} \psi(d(y_{2n}, y_{2n+1})) - \lim_{n \rightarrow \infty} \phi(N(x_{2n}, x_{2n+1})).$$

This implies that

$$\psi(r) \leq \psi(r) - \lim_{n \rightarrow \infty} \phi(N(x_{2n}, x_{2n+1})).$$

We observe that the last term on the right-hand side of the above inequality is non-zero. We get a contradiction with ϕ function. Therefore, we have

$$\lim_{n \rightarrow \infty} d(y_{2n}, y_{2n+1}) = 0.$$

Putting $x = x_{2n+1}$ and $y = x_{2n+2}$ in (2.1) and arguing as above, we obtain

$$\lim_{n \rightarrow \infty} d(y_{2n+1}, y_{2n+2}) = 0.$$

Therefore, for all $n \in N \cup \{0\}$, we have

$$\lim_{n \rightarrow \infty} d(y_n, y_{n+1}) = 0. \tag{2.18}$$

Next, we prove that $\{y_n\}$ is a Cauchy sequence. For this, it is enough to show that the subsequence $\{y_{2n}\}$ is a Cauchy sequence. To the contrary, suppose that $\{y_{2n}\}$ is not a Cauchy sequence, then there exist $\epsilon > 0$ and the sequence of natural numbers $\{2n(k)\}$ and $\{2m(k)\}$ such that $2n(k) > 2m(k) > 2k$ for $k \in N$ and

$$d(y_{2m(k)}, y_{2n(k)}) \geq \epsilon \tag{2.19}$$

corresponding to $2m(k)$. We can choose $2n(k)$ to be the smallest such that (2.19) is satisfied. Then we have

$$d(y_{2m(k)}, y_{2n(k)-1}) < \epsilon. \tag{2.20}$$

Putting $x = x_{2m(k)-1}$ and $y = x_{2n(k)-1}$ in (2.1), where for all $k \in N$,

$$\psi(d(y_{2m(k)}, y_{2n(k)})) \leq \psi(M(x_{2m(k)-1}, x_{2n(k)-1})) - \phi(N(x_{2m(k)-1}, x_{2n(k)-1})), \tag{2.21}$$

where

$$M(x_{2m(k)-1}, x_{2n(k)-1}) = \max \left\{ d(y_{2m(k)-1}, y_{2n(k)-1}), \right. \\ \left. \frac{1}{2} (d(y_{2m(k)-1}, y_{2m(k)}) + d(y_{2n(k)-1}, y_{2n(k)})), \right. \\ \left. \frac{1}{2} (d(y_{2m(k)-1}, y_{2n(k)}) + d(y_{2n(k)-1}, y_{2m(k)})) \right\}$$

and

$$N(x_{2m(k)-1}, x_{2n(k)-1}) = \min \left\{ d(y_{2m(k)-1}, y_{2n(k)-1}), \right. \\ \left. \frac{1}{2} (d(y_{2m(k)-1}, y_{2m(k)}) + d(y_{2n(k)-1}, y_{2n(k)})), \right. \\ \left. \frac{1}{2} (d(y_{2m(k)-1}, y_{2n(k)}) + d(y_{2n(k)-1}, y_{2m(k)})) \right\}.$$

Using the triangle inequality, we have

$$d(y_{2m(k)}, y_{2n(k)}) \leq d(y_{2m(k)}, y_{2m(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)}).$$

Taking the limit $k \rightarrow \infty$, we get

$$\lim_{k \rightarrow \infty} d(y_{2m(k)}, y_{2n(k)}) = \epsilon. \tag{2.22}$$

Again for all k , we have

$$d(y_{2m(k)-1}, y_{2n(k)-1}) \leq d(y_{2m(k)}, y_{2m(k)-1}) + d(y_{2m(k)}, y_{2n(k)}) + d(y_{2n(k)-1}, y_{2n(k)}),$$

$$d(y_{2m(k)}, y_{2n(k)}) \leq d(y_{2m(k)}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)}).$$

On letting limit $k \rightarrow \infty$ and using (2.18)-(2.22), we get

$$\lim_{k \rightarrow \infty} d(y_{2m(k)-1}, y_{2n(k)-1}) = \epsilon. \tag{2.23}$$

Again for all positive integers k , we have

$$d(y_{2m(k)-1}, y_{2n(k)}) \leq d(y_{2m(k)-1}, y_{2m(k)}) + d(y_{2m(k)}, y_{2n(k)}),$$

$$d(y_{2m(k)}, y_{2n(k)}) \leq d(y_{2m(k)}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2n(k)}).$$

Letting limit $k \rightarrow \infty$ and using (2.18)-(2.23), we get

$$\lim_{k \rightarrow \infty} d(y_{2m(k)-1}, y_{2n(k)}) = \epsilon. \tag{2.24}$$

Again for all positive integers k , we have

$$d(y_{2n(k)-1}, y_{2m(k)}) \leq d(y_{2n(k)-1}, y_{2n(k)}) + d(y_{2n(k)}, y_{2m(k)}),$$

$$d(y_{2n(k)}, y_{2m(k)}) \leq d(y_{2n(k)}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2m(k)}).$$

Letting limit $k \rightarrow \infty$ and using (2.18)-(2.24), we get

$$\lim_{k \rightarrow \infty} d(y_{2n(k)-1}, y_{2m(k)}) = \epsilon. \tag{2.25}$$

From (2.21)-(2.25), we get

$$\lim_{k \rightarrow \infty} M(x_{2m(k)-1}, x_{2n(k)-1}) = \epsilon$$

and

$$\lim_{k \rightarrow \infty} N(x_{2m(k)-1}, x_{2n(k)-1}) = 0.$$

Letting $k \rightarrow \infty$ in (2.21), we get

$$\psi(\epsilon) \leq \psi(\epsilon) - \lim_{k \rightarrow \infty} \phi(N(x_{2m(k)-1}, x_{2n(k)-1})).$$

Using discontinuity of ϕ at $t = 0$ and $\phi(t) > 0$ for $t > 0$, we observe that the last term on the right-hand side of the above inequality is non-zero. Thus we arrive at a contradiction.

Hence $\{y_n\}$ is a Cauchy sequence.

Therefore, the Cauchy sequence $\{y_n\}$ is a convergent sequence, and it converges to a point z (say) in X .

Consequently, the subsequences also converge to z in X :

$$Ax_{2n} \rightarrow z, \quad Tx_{2n+1} \rightarrow z, \quad Bx_{2n+1} \rightarrow z \quad \text{and} \quad Sx_{2n} \rightarrow z.$$

Now we shall prove that z is a common fixed point of A, B, S and T .

Since $B(X) \subset S(X)$, there exists $v \in X$ such that $z = Sv$. Let $d(z, Av) \neq 0$. Putting $x = v$ and $y = x_{2n+1}$ in (2.1), we get

$$\psi(d(Av, Bx_{2n+1})) \leq \psi(M(v, x_{2n+1}) - \phi(N(v, x_{2n+1})), \tag{2.26}$$

where

$$M(v, x_{2n+1}) = \max \left\{ d(Sv, Tx_{2n+1}), \frac{1}{2}(d(Sv, Av) + d(Tx_{2n+1}, Bx_{2n+1})), \right. \\ \left. \frac{1}{2}(d(Sv, Bx_{2n+1}) + d(Tx_{2n+1}Av)) \right\}$$

and

$$N(v, x_{2n+1}) = \min \left\{ d(Sv, Tx_{2n+1}), \frac{1}{2}(d(Sv, Av) + d(Tx_{2n+1}, Bx_{2n+1})), \right. \\ \left. \frac{1}{2}(d(Sv, Bx_{2n+1}) + d(Tx_{2n+1}, Av)) \right\}.$$

Taking $n \rightarrow \infty$ and using $z = Sv$, we have

$$M(v, z) = \max \left\{ d(Sv, z), \frac{1}{2}(d(Sv, Av) + d(z, z)), \frac{1}{2}(d(Sv, z) + d(z, Av)) \right\} = \frac{1}{2}(d(z, Av)).$$

Also we have

$$\psi(d(Av, z)) \leq \psi\left(\frac{1}{2}(d(z, Av))\right) - \lim_{n \rightarrow \infty} \phi(N(v, x_{2n+1})).$$

Using discontinuity of ϕ at $t = 0$ and $\phi(t) > 0$ for $t > 0$, we observe that the last term on the right-hand side of the above inequality is non-zero. Therefore we obtain

$$\psi(d(Av, z)) < \psi\left(\frac{1}{2}(d(z, Av))\right).$$

Hence we arrive at a contradiction with the ψ function.

Therefore $d(z, Av) = 0 \Rightarrow Av = z \Rightarrow Av = z = Sv$.

Since (A, S) is a weakly compatible pair of maps, so it commutes at their coincidence point v , i.e., $ASv = SAV \Rightarrow Az = Sz$.

Now we shall show that $Az = Sz = z$.

For this, putting $x = z$ and $y = x_{2n+1}$ in (2.1), we get

$$\psi(d(Az, Bx_{2n+1})) \leq \psi(M(z, x_{2n+1})) - \phi(N(z, x_{2n+1})), \tag{2.27}$$

where

$$M(z, x_{2n+1}) = \max\left\{d(Sz, Tx_{2n+1}), \frac{1}{2}(d(Sz, Az) + d(Tx_{2n+1}, Bx_{2n+1})), \frac{1}{2}(d(Sz, Bx_{2n+1}) + d(Tx_{2n+1}, Az))\right\}$$

and

$$N(z, x_{2n+1}) = \min\left\{d(Sz, Tx_{2n+1}), \frac{1}{2}(d(Sz, Az) + d(Tx_{2n+1}, Bx_{2n+1})), \frac{1}{2}(d(Sz, Bx_{2n+1}) + d(Tx_{2n+1}, Az))\right\}.$$

Taking $n \rightarrow \infty$ and using $Az = Sz$, we get

$$M(z, z) = d(Sz, z).$$

Now (2.27) implies that

$$\psi(d(Sz, z)) \leq \psi(d(Sz, z)) - \lim_{n \rightarrow \infty} \phi(N(z, x_{2n+1})).$$

Using discontinuity of ϕ at $t = 0$ and $\phi(t) > 0$ for $t > 0$, we observe that the last term on the right-hand side of the above inequality is non-zero. Therefore we obtain

$$\psi(d(Sz, z)) < \psi(d(Sz, z)),$$

which is contradiction. Therefore $d(Sz, z) = 0 \Rightarrow Sz = z \Rightarrow Sz = Az = z$. Similarly, we can show that $Tz = Bz = z$. Hence $Sz = Az = Tz = Bz = z$.

Now we shall show that z is the unique common fixed point of A, B, S and T .

Let z_1 be also a fixed point of A, B, S and T . Putting $x = z$ and $y = z_1$ in (2.1), we have

$$\psi(d(z, z_1)) \leq \psi(d(z, z_1)) - \phi(d(z, z_1)), \tag{2.28}$$

a contradiction. Hence $d(z, z_1) = 0 \Rightarrow z = z_1$. Hence A, B, S and T have a unique common fixed point in X . □

When we take $S = T = I$ identity map, we get the following theorem.

Theorem 2.2 *Let (X, d) be a complete metric space. Let $A, B : X \rightarrow X$ be two self-mappings which satisfy the following inequality:*

$$\psi(d(Ax, By)) \leq \psi(M(x, y)) - \phi(N(x, y)), \tag{2.29}$$

where $x, y \in X, x \neq y$,

$$M(x, y) = \max \left\{ d(x, y), \frac{1}{2}(d(x, Ax) + d(y, By)), \frac{1}{2}(d(x, By) + d(y, Ax)) \right\}$$

and

$$N(x, y) = \min \left\{ d(x, y), \frac{1}{2}(d(x, Ax) + d(y, By)), \frac{1}{2}(d(x, By) + d(y, Ax)) \right\};$$

- (1) $\phi : [0, \infty) \rightarrow [0, \infty)$ is such that $\phi(t) > 0$ which is lower semi-continuous for all $t > 0$, and ϕ is discontinuous at $t = 0$ with $\phi(0) = 0$,
- (2) $\psi : [0, \infty) \rightarrow [0, \infty)$ is an altering distance function.

Then there exists a unique fixed point of A, B in X .

When we take $\psi(t) = t$ in Theorem 2.1 and Theorem 2.2, we get the following corollaries.

Corollary 2.3 *Let (X, d) be a complete metric space. Let A, B, S and $T : X \rightarrow X$ be self-mappings which satisfy the following inequality:*

$$d(Ax, By) \leq M(x, y) - \phi(N(x, y)), \tag{2.30}$$

where $x, y \in X, x \neq y$ and

$$M(x, y) = \max \left\{ d(Sx, Ty), \frac{1}{2}(d(Sx, Ax) + d(Ty, By)), \frac{1}{2}(d(Sx, By) + d(Ty, Ax)) \right\},$$

$$N(x, y) = \min \left\{ d(Sx, Ty), \frac{1}{2}(d(Sx, Ax) + d(Ty, By)), \frac{1}{2}(d(Sx, By) + d(Ty, Ax)) \right\};$$

- (1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,
- (2) (A, S) and (B, T) are weak compatible pairs,
- (3) $\phi : [0, \infty) \rightarrow [0, \infty)$ is such that $\phi(t) > 0$ which is lower semi-continuous for all $t > 0$, and ϕ is discontinuous at $t = 0$ with $\phi(0) = 0$.

Then A, B, S and T have a unique common fixed point in X .

Corollary 2.4 *Let (X, d) be a complete metric space. Let $A, B : X \rightarrow X$ be two self-mappings which satisfy the following inequality:*

$$d(Ax, By) \leq M(x, y) - \phi(N(x, y)), \tag{2.31}$$

where $x, y \in X, x \neq y,$

$$M(x, y) = \max \left\{ d(x, y), \frac{1}{2}(d(x, Ax) + d(y, By)), \frac{1}{2}(d(x, By) + d(y, Ax)) \right\} \text{ and} \tag{2.32}$$

$$N(x, y) = \min \left\{ d(x, y), \frac{1}{2}(d(x, Ax) + d(y, By)), \frac{1}{2}(d(x, By) + d(y, Ax)) \right\};$$

$\phi : [0, \infty) \rightarrow [0, \infty)$ is such that $\phi(t) > 0$ which is lower semi-continuous for all $t > 0,$ and ϕ is discontinuous at $t = 0$ with $\phi(0) = 0.$

Then there exists a unique fixed point of A, B in $X.$

Now, we are ready to establish the following theorem in which we are going to replace

$$N(x, y) = \min \left\{ d(Sx, Ty), \frac{1}{2}(d(Sx, Ax) + d(Ty, By)), \frac{1}{2}(d(Sx, By) + d(Ty, Ax)) \right\}$$

with

$$N(x, y) = \min \left\{ d(Sx, Ty), \frac{1}{2}(d(Sx, By) + d(Ty, Ax)) \right\}.$$

Theorem 2.5 *Let (X, d) be a complete metric space. Let A, B, S and $T : X \rightarrow X$ be self-mappings which satisfy the following inequality:*

$$\psi(d(Ax, By)) \leq \psi(M(x, y)) - \phi(N(x, y)) \tag{2.33}$$

for all $x, y \in X$ with $x \neq y$ and

$$M(x, y) = \max \left\{ d(Sx, Ty), \frac{1}{2}(d(Sx, Ax) + d(Ty, By)), \frac{1}{2}(d(Sx, By) + d(Ty, Ax)) \right\}$$

and

$$N(x, y) = \min \left\{ d(Sx, Ty), \frac{1}{2}(d(Sx, By) + d(Ty, Ax)) \right\};$$

- (1) $A(X) \subset T(X)$ and $B(X) \subset S(X),$
- (2) (A, S) and (B, T) are weak compatible pairs,
- (3) $\phi : [0, \infty) \rightarrow [0, \infty)$ is a lower semi-continuous function with $\phi(t) > 0$ for all $t \in (0, \infty)$ and $\phi(0) = 0,$
- (4) $\psi : [0, \infty) \rightarrow [0, \infty)$ is an altering distance function which in addition is strictly monotone increasing.

Then A, B, S and T have a unique common fixed point in $X.$

One can easily prove this theorem by using the technique given in the proof of Theorem 2.1.

When we take $S = T = I$ identity map, we get the following theorem.

Theorem 2.6 *Let (X, d) be a complete metric space. Let A and $B : X \rightarrow X$ be self-mappings which satisfy the following inequality:*

$$\psi(d(Ax, By)) \leq \psi(M(x, y)) - \phi(N(x, y)) \tag{2.34}$$

for all $x, y \in X$ with $x \neq y$,

$$M(x, y) = \max \left\{ d(x, y), \frac{1}{2}(d(x, Ax) + d(Ty, By)), \frac{1}{2}(d(x, By) + d(y, Ax)) \right\}$$

and

$$N(x, y) = \min \left\{ d(x, y), \frac{1}{2}(d(x, By) + d(y, Ax)) \right\};$$

- (1) $\phi : [0, \infty) \rightarrow [0, \infty)$ is a lower semi-continuous function with $\phi(t) > 0$ for all $t \in (0, \infty)$ and $\phi(0) = 0$,
- (2) $\psi : [0, \infty) \rightarrow [0, \infty)$ is an altering distance function which in addition is strictly monotone increasing.

Then there exists a unique fixed point of A, B in X .

When we take $\psi(t) = t$ in Theorem 2.5 and Theorem 2.6, we get the following corollaries.

Corollary 2.7 *Let (X, d) be a complete metric space. Let A, B, S and $T : X \rightarrow X$ be self-mappings which satisfy the following inequality:*

$$d(Ax, By) \leq M(x, y) - \phi(N(x, y)), \tag{2.35}$$

where $x, y \in X, x \neq y$ and

$$M(x, y) = \max \left\{ d(Sx, Ty), \frac{1}{2}(d(Sx, Ax) + d(Ty, By)), \frac{1}{2}(d(Sx, By) + d(Ty, Ax)) \right\},$$

$$N(x, y) = \min \left\{ d(Sx, Ty), \frac{1}{2}(d(Sx, By) + d(Ty, Ax)) \right\};$$

- (1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,
- (2) (A, S) and (B, T) are weak compatible pairs,
- (3) $\phi : [0, \infty) \rightarrow [0, \infty)$ is a lower semi-continuous function with $\phi(t) > 0$ for all $t \in (0, \infty)$ and $\phi(0) = 0$.

Then A, B, S and T have a unique common fixed point in X .

Corollary 2.8 *Let (X, d) be a complete metric space. Let $A, B : X \rightarrow X$ be two self-mappings which satisfy the following inequality:*

$$d(Ax, By) \leq M(x, y) - \phi(N(x, y)) \tag{2.36}$$

for all $x, y \in X$ with $x \neq y$,

$$M(x, y) = \max \left\{ d(Sx, Ty), \frac{1}{2}(d(Sx, Ax) + d(Ty, By)), \frac{1}{2}(d(Sx, By) + d(Ty, Ax)) \right\}$$

and

$$N(x, y) = \min \left\{ d(Sx, Ty), \frac{1}{2}(d(Sx, By) + d(Ty, Ax)) \right\};$$

$\phi : [0, \infty) \rightarrow [0, \infty)$ is a lower semi-continuous function with $\phi(t) > 0$ for all $t \in (0, \infty)$ and $\phi(0) = 0$.

Then there exists a unique fixed point of A, B in X .

Example 2.9 Let $X = [0, 2]$ be endowed with the Euclidean metric $d(x, y) = |x - y|$, and let A, B, S and $T \rightarrow X$ be defined by

$$A(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{x}{5} + 1 & \text{otherwise,} \end{cases} \quad S(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{3x}{5} + 1 & \text{otherwise,} \end{cases}$$

$$B(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{2x}{5} + 1 & \text{otherwise,} \end{cases} \quad T(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{4x}{5} + 1 & \text{otherwise,} \end{cases}$$

where $x, y \in X$. $A(X) = [0, \frac{7}{5}]$, $S(X) = [0, \frac{11}{5}]$, $B(X) = [0, \frac{9}{5}]$, $T(X) = [0, \frac{13}{5}]$. Here $A(X) \subset T(X)$ and $B(X) \subset S(X)$, and (A, S) and (B, T) are weakly compatible maps at $x = 0$ but not compatible maps.

Take

$$\psi(t) = t \quad \text{and} \quad \phi(t) = \begin{cases} \frac{t}{2} & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Now we have to check the inequality of Theorem 2.1 for the following cases.

Case 1: If $x = 0$ and $y = 0$:

$$\psi(d(Ax, By)) = \psi(|Ax - By|) = 0, \quad \psi(M(x, y)) = 0 \quad \text{and} \quad \phi(N(x, y)) = 0,$$

hence $\psi(d(Ax, By)) = \psi(M(x, y)) - \phi(N(x, y))$.

Case 2: If $x = 0$ and $y \neq 0$:

$$\psi(d(Ax, By)) = \psi(|Ax - By|) = \psi\left(\left|\frac{2y}{5}\right|\right) = \left|\frac{2y}{5}\right|$$

and

$$M(x, y) = \max \left[|Sx - Ty|, \frac{1}{2}(|Sx - Ax| + |Ty - By|), \frac{1}{2}(|Sx - By| + |Ty - Ax|) \right]$$

$$= \max \left[\left|\frac{4y}{5}\right|, \left|\frac{y}{5}\right|, \frac{1}{2} \left(\left|\frac{2y}{5}\right| + \left|\frac{4y}{5}\right| \right) \right] = \left|\frac{4y}{5}\right|,$$

$$N(x, y) = \min \left[|Sx - Ty|, \frac{1}{2}(|Sx - Ax| + |Ty - By|), \frac{1}{2}(|Sx - By| + |Ty - Ax|) \right]$$

$$\begin{aligned}
 &= \min \left[\left| \frac{4y}{5} \right|, \left| \frac{y}{5} \right|, \frac{1}{2} \left(\left| \frac{2y}{5} \right| + \left| \frac{4y}{5} \right| \right) \right] = \left| \frac{y}{5} \right|, \\
 \psi(M(x, y)) - \Phi(N(x, y)) &= \psi \left(\left| \frac{4y}{5} \right| \right) - \phi \left(\left| \frac{y}{5} \right| \right) \\
 &= \left| \frac{4y}{5} \right| - \frac{1}{2} \left(\left| \frac{y}{5} \right| \right) \\
 &\geq \left| \frac{2y}{5} \right| = \psi(d(Ax, By)).
 \end{aligned}$$

Case 3: If $x \neq 0$ and $y = 0$:

$$\psi(d(Ax, By)) = \psi(|Ax - By|) = \psi(|Ax|) = \psi \left(\left| \frac{x}{5} \right| \right) = \left| \frac{x}{5} \right|$$

and

$$\begin{aligned}
 M(x, y) &= \max \left[|Sx - Ty|, \frac{1}{2}(|Sx - Ax| + |Ty - By|), \frac{1}{2}(|Sx - By| + |Ty - Ax|) \right] \\
 &= \max \left[\left| \frac{3x}{5} \right|, \left| \frac{x}{5} \right|, \frac{1}{2} \left(\left| \frac{3x}{5} \right| + \left| \frac{x}{5} \right| \right) \right] = \left| \frac{3x}{5} \right|, \\
 N(x, y) &= \min \left[|Sx - Ty|, \frac{1}{2}(|Sx - Ax| + |Ty - By|), \frac{1}{2}(|Sx - By| + |Ty - Ax|) \right] \\
 &= \min \left[\left| \frac{3x}{5} \right|, \left| \frac{x}{5} \right|, \frac{1}{2} \left(\left| \frac{3x}{5} \right| + \left| \frac{x}{5} \right| \right) \right] = \left| \frac{x}{5} \right|, \\
 \psi(M(x, y)) - \phi(N(x, y)) &= \psi \left(\left| \frac{3x}{5} \right| \right) - \phi \left(\left| \frac{x}{5} \right| \right) \\
 &= \left| \frac{3x}{5} \right| - \frac{1}{2} \left(\left| \frac{x}{5} \right| \right) \\
 &\geq \left| \frac{x}{5} \right| = \psi(d(Ax, By)).
 \end{aligned}$$

Case 4: If $x \neq 0$ and $y \neq 0$:

$$\psi(d(Ax, By)) = \psi(|Ax - By|) = \psi \left(\left| \frac{x}{5} + 1 - \frac{2y}{5} - 1 \right| \right) = \left| \frac{x}{5} - \frac{2y}{5} \right|$$

and

$$\begin{aligned}
 M(x, y) &= \max \left[|Sx - Ty|, \frac{1}{2}(|Sx - Ax| + |Ty - By|), \frac{1}{2}(|Sx - By| + |Ty - Ax|) \right] \\
 &= \max \left[\left| \frac{3x}{5} - \frac{4y}{5} \right|, \left(\left| \frac{x}{5} \right| + \left| \frac{y}{5} \right| \right), \frac{1}{2} \left(\left| \frac{3x}{5} - \frac{2y}{5} \right| + \left| \frac{4y}{5} - \frac{x}{5} \right| \right) \right], \\
 M(x, y) &= \begin{cases} \left| \frac{3x}{5} - \frac{4y}{5} \right| & \text{if } 0 < y < \frac{2x}{5}, \\ \left| \frac{3x}{5} - \frac{4y}{5} \right|, \left(\left| \frac{x}{5} \right| + \left| \frac{y}{5} \right| \right), \frac{1}{2} \left(\left| \frac{3x}{5} - \frac{2y}{5} \right| + \left| \frac{4y}{5} - \frac{x}{5} \right| \right) & \text{if } y = \frac{2x}{5}, \\ \left(\left| \frac{x}{5} \right| + \left| \frac{y}{5} \right| \right), \frac{1}{2} \left(\left| \frac{3x}{5} - \frac{2y}{5} \right| + \left| \frac{4y}{5} - \frac{x}{5} \right| \right) & \text{if } \frac{2x}{5} < y \leq x, \end{cases} \\
 N(x, y) &= \min \left[|Sx - Ty|, \frac{1}{2}(|Sx - Ax| + |Ty - By|), \frac{1}{2}(|Sx - By| + |Ty - Ax|) \right]
 \end{aligned}$$

$$= \min \left[\left| \frac{3x}{5} - \frac{4y}{5} \right|, \left(\left| \frac{x}{5} \right| + \left| \frac{y}{5} \right| \right), \frac{1}{2} \left(\left| \frac{3x}{5} - \frac{2y}{5} \right| + \left| \frac{4y}{5} - \frac{x}{5} \right| \right) \right],$$

$$N(x, y) = \begin{cases} \left| \frac{x}{5} \right| + \left| \frac{y}{5} \right| & \text{if } 0 < y \leq \frac{x}{5}, \\ \left| \frac{x}{5} \right| + \left| \frac{y}{5} \right|, \frac{1}{2} \left(\left| \frac{3x}{5} - \frac{2y}{5} \right| + \left| \frac{4y}{5} - \frac{x}{5} \right| \right) & \text{if } \frac{x}{5} < y < \frac{2x}{5}, \\ \left| \frac{3x}{5} - \frac{4y}{5} \right|, \left| \frac{x}{5} \right| + \left| \frac{y}{5} \right|, \frac{1}{2} \left(\left| \frac{3x}{5} - \frac{2y}{5} \right| + \left| \frac{4y}{5} - \frac{x}{5} \right| \right) & \text{if } y = \frac{2x}{5}, \\ \left| \frac{3x}{5} - \frac{4y}{5} \right| & \text{if } \frac{2x}{5} < y \leq x, \end{cases}$$

$$\psi(d(Ax, By)) = \left| \frac{x}{5} - \frac{2y}{5} \right| \leq \psi(M(x, y)) - \phi(N(x, y)).$$

So the inequalities hold in each of the cases. Hence all the conditions of Theorem 2.1 hold and A , B , S and T have the unique common fixed point at $x = 0$ in X .

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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