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General solution and generalized Hyers-Ulam stability for additive functional equations

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Abstract

In this paper, we introduce new types of additive functional equations and obtain the solutions to these additive functional equations. Furthermore, we investigate the Hyers-Ulam stability for the additive functional equations in fuzzy normed spaces and random normed spaces using the direct and fixed point approaches. Also, we will present some applications of functional equations in physics. Through these examples, we explain how the functional equations appear in the physical problem, how we use them to solve it, and we talk about solutions that are not used for solving the problem, but which can be of interest. We provide an example to show how functional equations may be used to solve geometry difficulties.

Keywords: Hyers-Ulam stability, fuzzy normed space, random normed space and fixed point.

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1. Introduction

A classical question in the theory of functional equations is: when is it true that a mapping, which approximately satisfying a functional equation, must be somehow close to an exact solution of the equation? Such a problem, called a stability problem of the functional equation, was formulated by Ulam [44]

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and Hyers [14] and they have gave a solution to Ulam's problem for the case of approximate additive mappings. Subsequently, Hyers result was generalized by Aoki [2] for additive mappings and by Rassias [37] for linear mappings to consider the problem with the unbounded Cauchy differences. The stability problems of functional equations have been extensively investigated by [1, 4, 6, 9–20].

Katsaras [17] defined a fuzzy norm on a linear space to construct a fuzzy structure on the space. Some mathematicians have introduced several types of fuzzy norms from different points of view. In particular, Bag and Samanta [3] and Cheng et al. [5] have gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of the Kramosil and Michalek type [21]. In [30], the authors have introduced the notion of fuzzy Hyers-Ulam-Rassias stability. These genuine foundations serve as the basis for the Hyers-Ulam-Aoki-Rassias stability (see [7, 8, 40]). In [38, 39], the authors have gave speculations on the Hyers stability hypothesis, which permits the Cauchy difference to be bounded. Radu [36] investigated the stability of functional equations by using the fixed point method.

A mapping, which fulfills

$$\Psi(F+G) = \Psi(F) + \Psi(G) \tag{1.1}$$

is called an additive mapping.

Kannappan [16] investigated the stability of various types of functional equations, such as additive equations, multiplicative equations, logarithmic functions, trigonometric functions, vector-valued functions, sine functional equations, alternative Cauchy equations, wave equations, polynomial equations, and quadratic equations. Sahoo and Kannappan [41], presented Hyers result along with a theorem due to Rassias that generalizes the result of Hyers. We also point out some other generalizations related to the stability of the additive Cauchy equation.

The importance of functional equations is comparable to that of differential equations because many of the problems that can be stated in terms of a differential equation or a system of differential equations can also be stated in terms of a functional equation or a system of functional equations. However, functional equations are easier and more natural than differential equations because they do not involve derivatives.

An exploration of recent research with Choquet-Deny-type equations and characterization theory, which demonstrates the close links that have been established between stochastic processes and applied probability. Concentrates on solutions of the Integrated Cauchy Functional Equation (ICFE) under different conditions. Demonstrates that results from a variety of characterization problems involving discrete and continuous distributions can be more easily obtained by utilizing the solution of an appropriate ICFE. Discussion of applications of the ICFE in characterizing stochastic models is included, with additional examples from areas such as renewal processes and potential theory. For more details of applications of functional equations and functional differential equations, see [22–35, 41–45].

In this paper, we investigate general solution and generalized Hyers-Ulam stability of the functional equations

$$\Psi\left(\sum_{p=1}^{\phi}\left(\frac{\mathfrak{c}+\mathfrak{e}}{2}\right)^{\phi} h_p\right) - \left(\frac{\mathfrak{c}+\mathfrak{e}}{2}\right)^{\phi}\sum_{p=1}^{\phi}\Psi(h_p) = 0 \tag{1.2}$$

and

$$\Psi\left(\sum_{p=1}^{\phi} h_p\right) + \sum_{q=1}^{\phi}\Psi\left(-h_q + \sum_{p=1;p\neq q} h_p\right) - (\phi-1)\sum_{p=1}^{\phi}\Psi(h_p) = 0, \tag{1.3}$$

where $\mathfrak{c} \neq \mathfrak{e} \neq 0$ and $\phi > 1$ in fuzzy normed space and random normed space as well as using two different methods.

2. Preliminaries

In this section, we introduce the definition of a fuzzy normed space and a random normed space.

Definition 2.1. Let Θ be a real linear space. A function $\mho : \Theta \times \mathcal{R} \to [0,1]$ is said to be fuzzy norm on Θ if

N_1. $\mho(h,\mathfrak{c}) = 0$ for $\mathfrak{c} \leqslant 0$;
N_2. $h = 0$ iff $\mho(h,\mathfrak{c}) = 1,\ \forall\ \mathfrak{c} > 0$;
N_3. $\mho(ch,\mathfrak{e}) = \mho\left(h, \frac{\mathfrak{e}}{|c|}\right)$ if $c \neq 0$;
N_4. $\mho(h+\mathfrak{g},\mathfrak{c}+\mathfrak{e}) \geqslant \min\{\mho(h,\mathfrak{e}),\mho(\mathfrak{g},\mathfrak{e})\}$;
N_5. $\mho(h,.)$ is a increasing on $\mathcal{R}$ and $\lim_{\mathfrak{c}\to\infty} \mho(h,\mathfrak{c}) = 1$;
N_6. for $h \neq 0, \mho(h,.)$ is continuous on $\mathcal{R}$, $\forall\ h,\mathfrak{g} \in \Theta$ and $\mathfrak{c},\mathfrak{e} \in \mathcal{R}$.

The pair $(\Theta,\mho)$ is called "fuzzy normed space."

Definition 2.2. Let $\mathbb{T}$ be a continuous t-norm, Θ-vector space and a mapping $\Upsilon : \Theta \to D^+$. A random normed space is a triple $(\Theta, \Upsilon, \mathbb{T})$, if

1. $\Upsilon_h(s) = \epsilon_0(s)$, for all $s > 0 \Leftrightarrow h = 0$;
2. $\Upsilon_{\alpha h}(s) = \Upsilon_\alpha\left(\frac{s}{|\alpha|}\right)$, for all $h \in \Theta$, and α in $\mathcal{R}$ with $\alpha \neq 0$;
3. $\Upsilon_{h+\mathfrak{g}}(s+\iota) \geqslant (\Upsilon_h(s);\Upsilon_y(s))$, for all $h,\mathfrak{g} \in \Theta$ and $s,\iota \geqslant 0$.

Theorem 2.3. *Let* $(\mho, d)$ *be a complete metric space and* $\Omega : \mho \to \mathbb{Z}$ *be a mapping with Lipschitz constant* $\pounds$*. Then for each given element* $h \in \mho$*, we have*

1. $(\Omega^\phi h, \Omega^{\phi+1} h) = +\infty,\ \ \forall\ \phi \geqslant 0$;
2. $d(\Omega^\phi h, \Omega^{\phi+1} h) < \infty,\ \ \forall\ \phi \geqslant \phi_0$;
3. $\{\Omega^\phi h\}$ *is convergent to* z^* *of* Ω;
4. $\mathbb{Z} = \{z \in \mho; d(\Omega^{\phi_0} h, z) < \infty\}$ *for* z^* *of* Ω;
5. $\frac{1}{1-\pounds} d(Q,\Omega Q) \geqslant d(Q^*,Q),\ \forall\ Q \in \pounds$.

Here, Ω *is a strictly contractive mapping.*

Throughout this paper, an additive function $\Psi : \Theta \to \mathcal{Y}$ is defined by

$$D\Psi(h_1,h_2,\ldots,h_\phi) = \Psi\left(\sum_{p=1}^{\phi}\left(\frac{\mathfrak{c}+\mathfrak{e}}{2}\right)^{\phi} h_p\right) - \left(\frac{\mathfrak{c}+\mathfrak{e}}{2}\right)^{\phi}\sum_{p=1}^{\phi}\Psi(h_p)$$

and

$$D\Psi(h_1,h_2,\ldots,h_\phi) = \Psi\left(\sum_{p=1}^{\phi} h_p\right) + \sum_{q=1}^{\phi}\Psi\left(-h_q + \sum_{p=1;p\neq q} h_p\right) - (\phi-1)\sum_{p=1}^{\phi}\Psi(h_p),$$

for all $h_1,h_2,\ldots,h_\phi \in \Theta$.

3. General solution of (1.2)

In this section, we present the general solution of the ϕ-dimensional functional equation (1.2).

Theorem 3.1. *Let* $(\Theta,\mathcal{Y})$ *be a real vector space. If a mapping* $\Psi : \Theta \to \mathcal{Y}$ *satisfies* (1.3)*, for all* $h_1,h_2,\ldots,h_\phi \in \Theta$*, then* $\Psi : \Theta \to \mathcal{Y}$ *satisfies* (1.1)*, for all* $F,G \in \Theta$.

Proof. Suppose that a mapping $\Psi : \Theta \to \mathcal{Y}$ satisfies the functional equation (1.1). Setting (F, G) by $(0, 0)$ in (1.1), we get $\Psi(0) = 0$. Replacing (F, G) by (h_1, h_2) in (1.1), we have

$$\Psi(h_1 + h_2) = \Psi(h_1) + \Psi(h_2), \ \forall \ h_1, h_2 \in \Theta. \tag{3.1}$$

Setting (h_1, h_2) by $(h_1 - h)$ in (3.1), we get

$$\Psi(-h) = -\Psi(h), \ \forall \ h \in \Theta.$$

Therefore, Ψ is an odd function. Replacing (h_1, h_2) by (h, h) and $(2h, h)$ in (3.1), we arrive

$$\Psi(2h) = 2\Psi(h), \quad \Psi(3h) = 3\Psi(h), \ \forall \ h \in \Theta.$$

In general, for any positive integer ϕ, we have

$$\Psi(\phi h) = \phi\Psi(h), \ \forall \ h \in \Theta. \tag{3.2}$$

Switching (h_1, h_2) by $(h_1, -h_2)$ in (3.2), we get

$$\Psi(h_1 - h_2) = \Psi(h_1) - \Psi(h_2), \ \forall \ h_1, h_2 \in \Theta. \tag{3.3}$$

Setting (h_1, h_2) by $(-h_1, h_2)$ in (3.3), we have

$$\Psi(-h_1 + h_2) = -\Psi(h_1) + \Psi(h_2), \ \forall \ h_1, h_2 \in \Theta. \tag{3.4}$$

Adding (3.1), (3.3), and (3.4), we arrive

$$\Psi(h_1 + h_2) + \Psi(h_1 - h_2) + \Psi(-h_1 + h_2) = \Psi(h_1) + \Psi(h_2), \ \forall \ h_1, h_2 \in \Theta.$$

Substituting (h_1, h_2) by $(h_1, h_2 + h_3)$ in (3.1), we get

$$\Psi(h_1 + h_2 + h_3) = \Psi(h_1) + \Psi(h_2) + \Psi(h_3), \ \forall \ h_1, h_2, h_3 \in \Theta. \tag{3.5}$$

Setting (h_1, h_2) by $(-h_1, h_2 + h_3)$ in (3.1), we arrive

$$\Psi(-h_1 + h_2 + h_3) = -\Psi(h_1) + \Psi(h_2) + \Psi(h_3), \ \forall \ h_1, h_2, h_3 \in \Theta. \tag{3.6}$$

Replacing (h_1, h_2) by $(h_1, -h_2 + h_3)$ in (3.1), we get

$$\Psi(h_1 - h_2 + h_3) = \Psi(h_1) - \Psi(h_2) + \Psi(h_3), \ \forall \ h_1, h_2, h_3 \in \Theta. \tag{3.7}$$

Setting (h_1, h_2) by $(h_1, h_2 - h_3)$ in (3.1), we get

$$\Psi(h_1 + h_2 - h_3) = \Psi(h_1) + \Psi(h_2) - \Psi(h_3), \ \forall \ h_1, h_2, h_3 \in \Theta. \tag{3.8}$$

Adding (3.5), (3.6), (3.7), and (3.8), we arrive

$$\Psi(h_1 + h_2 + h_3) + \Psi(-h_1 + h_2 + h_3) + \Psi(h_1 - h_2 + h_3) + \Psi(h_1 + h_2 - h_3) = 2\Psi(h_1) + 2\Psi(h_2) + 2\Psi(h_3),$$

$\forall \ h_1, h_2, h_3 \in \Theta$. Continue this process up to ϕ times, we get

$$\Psi\left(\sum_{p=1}^{\phi} h_p\right) + \sum_{q=1}^{\phi} \Psi\left(-h_q + \sum_{p=1; p\neq q} h_p\right) - (\phi - 1)\sum_{p=1}^{\phi} \Psi(h_p) = 0, \ \forall \ h_1, h_2, h_3, \dots, h_\phi \in \Theta.$$

Conversely, a mapping $\Psi : \Theta \to \mathcal{Y}$ satisfies the functional equation (1.3). Replacing $(h_1, h_2, h_3, \dots, h_\phi)$ by $(0, 0, 0, \dots, 0)$ in (1.3), we get $\Psi(0) = 0$. Setting $(h_1, h_2, \dots, h_\phi)$ by $(h, 0, \dots, 0)$ in (1.3), we get

$$\phi\Psi(h) + \Psi(-h) = (\phi - 1)\Psi(h), \ \forall\ h \in \Theta. \tag{3.9}$$

From (3.9), we get

$$\Psi(-h) = -\Psi(h), \ \forall\ h \in \Theta.$$

Therefore, Ψ is an odd function. Setting $(h_1, h_2, h_3, \dots, h_\phi)$ by $(h, h, 0, \dots, 0)$, $(h, h, h, \dots, 0)$ in (1.3), we have

$$\Psi(2h) = 2\Psi(h), \ \ \Psi(3h) = 3\Psi(h), \ \forall\ h \in \Theta.$$

Switching $(h_1, h_2, h_3, \dots, h_\phi)$ by $(h_1, h_2, 0, \dots, 0)$ in (1.3), we get

$$\begin{aligned}\Psi(h_1 + h_2) &= (\phi - 1)\Psi(h_1) + \Psi(h_2),\\ 2\Psi(h_1 + h_2) &= (\phi - 1)\Psi(h_1) + \Psi(h_2),\\ 3\Psi(h_1 + h_2) &= (\phi - 1)\Psi(h_1) + \Psi(h_2), \ \forall\ h_1, h_2 \in \Theta.\end{aligned}$$

Continue this process up to ϕ times, we get

$$\Psi(h_1 + h_2) = \Psi(h_1) + \Psi(h_2), \ \forall\ h_1, h_2 \in \Theta.$$

Replacing (h_1, h_2) by (F, G), we have

$$\Psi(F + G) = \Psi(F) + \Psi(G), \ \forall\ F, G \in \Theta.$$

□

4. Direct method-Stability of (1.2)

In this section, we investigate the Hyers-Ulam stability of (1.2) in fuzzy normed space via direct method.

Theorem 4.1. *Let* $\Omega \in \{-1, 1\}$ *and* $\Gamma : \Theta^\phi \to \mathbb{Z}$ *be a mapping. Then*

$$\mathcal{N}'(\Gamma(\mathfrak{c}^\Omega h, \mathfrak{c}^\Omega h, \dots, \mathfrak{c}^\Omega h), \in) \geqslant \mathcal{N}'(\sigma^\Omega \Gamma(h, h, \dots, h), \epsilon), \ \sigma > 0, \left(\frac{\sigma}{\mathfrak{c}}\right)^\Omega < 1, \ \forall\ h \in \Theta, \ \epsilon > 0 \tag{4.1}$$

and

$$\lim_{\phi \to \infty} \mathcal{N}'(\Gamma(\mathfrak{c}^{\Omega\phi} h_1, \mathfrak{c}^{\Omega\phi} h_2, \mathfrak{c}^{\Omega\phi} h_3, \dots, h_\phi), \mathfrak{c}^{\Omega\phi}\epsilon) = 1, \ \forall\ h_1, h_2, \dots, h_\phi \in \Theta, \ \epsilon > 0.$$

Suppose an odd mapping $\Psi : \Theta \to \mathcal{Y}$ *fulfills the inequality*

$$\mathcal{N}(D\Psi(h_1, h_2, \dots, h_\phi), \epsilon) \geqslant \mathcal{N}'(\Gamma(h_1, h_2, \dots, h_\phi), \epsilon), \ \forall\ \epsilon > 0, \ h_1, h_2, \dots, h_\phi \in \Theta. \tag{4.2}$$

Then

$$A(h) = \mathcal{N} - \lim_{\phi \to \infty} \frac{\Psi(\mathfrak{c}^{\Omega\phi} h)}{\mathfrak{c}^{\Omega\phi}} \ \textit{exists, for all}\ h \in \Theta$$

and $A : \Theta \to \mathcal{Y}$ *is a unique additive mapping such that*

$$\mathcal{N}(\Psi(h) - A(h), \epsilon) \geqslant \mathcal{N}'(\Gamma(h, h, \dots, h), \epsilon|\mathfrak{c} - \sigma|), \ \forall\ h \in \Theta, \ \epsilon > 0, \tag{4.3}$$

where $\mathfrak{c} = \phi(\phi^2 - 2\phi + 1)$.

Proof. First assume $\Omega = 1$. Replacing $(h_1, h_2, \ldots, h_\phi)$ by $(h, h, \ldots, h)$ in (4.2), we obtain

$$\mathcal{N}(\Psi(\mathfrak{c}h) - \mathfrak{c}\Psi(h), \epsilon) \geqslant \mathcal{N}'(\Gamma(h, h, \ldots, h), \epsilon),\ \forall\, h \in \Theta,\ \epsilon > 0. \tag{4.4}$$

From (4.4), we arrive that

$$\mathcal{N}\left(\frac{\Psi(\mathfrak{c}h)}{\mathfrak{c}} - \Psi(h), \frac{\epsilon}{\mathfrak{c}}\right) \geqslant \mathcal{N}'(\Gamma(h, h, \ldots, h), \epsilon),\ \forall\, h \in \Theta,\ \epsilon > 0. \tag{4.5}$$

Replacing h by $\mathfrak{c}^\phi h$ in (4.5), we obtain

$$\mathcal{N}\left(\frac{\Psi(\mathfrak{c}^{\phi+1}h)}{\mathfrak{c}} - \Psi(\mathfrak{c}^\phi h), \frac{\epsilon}{\mathfrak{c}^{(\mathfrak{c})}}\right) \geqslant \mathcal{N}'(\Gamma(\mathfrak{c}^\phi h, \mathfrak{c}^\phi h, \ldots, \mathfrak{c}^\phi h), \epsilon),\ \forall\, h \in \Theta,\ \epsilon > 0. \tag{4.6}$$

Using (4.1), (N_3), in (4.6), we get

$$\mathcal{N}\left(\frac{\Psi(\mathfrak{c}^{\phi+1}h)}{\mathfrak{c}} - \Psi(\mathfrak{c}^\phi h), \frac{\epsilon}{\mathfrak{c}^{(\mathfrak{c})}}\right) \geqslant \mathcal{N}'\left(\Gamma(h, h, \ldots, h), \frac{\epsilon}{\sigma^\phi}\right),$$
$$\mathcal{N}\left(\frac{\Psi(\mathfrak{c}^{(\phi+1)}h)}{\mathfrak{c}^{(\phi+1)}} - \frac{\Psi(\mathfrak{c}^\phi h)}{\mathfrak{c}^\phi}, \frac{\epsilon}{\mathfrak{c}^{\mathfrak{c}}(\mathfrak{c}^\phi)}\right) \geqslant \mathcal{N}'\left(\Gamma(h, h, \ldots, h), \frac{\epsilon}{\sigma^\phi}\right),\ \forall\, h \in \Theta,\ \epsilon > 0. \tag{4.7}$$

Replacing ϵ by $\sigma^\phi \epsilon$ in (4.7), we get

$$\mathcal{N}\left(\frac{\Psi(\mathfrak{c}^{\phi+1}h)}{\mathfrak{c}^{(\phi+1)}} - \frac{\Psi(\mathfrak{c}^\phi h)}{\mathfrak{c}^\phi}, \frac{\sigma^\phi \epsilon}{\mathfrak{c}^{\mathfrak{c}}(\mathfrak{c}^\phi)}\right) \geqslant \mathcal{N}'\left(\Gamma(h, h, \ldots, h), \epsilon\right),\ \forall\, h \in \Theta,\ \epsilon > 0. \tag{4.8}$$

Clearly,

$$\frac{\Psi(\mathfrak{c}^\phi x)}{\mathfrak{c}^\phi} - \Psi(h) = \sum_{p=0}^{\phi-1} \frac{\Psi(\mathfrak{c}^{p+1}h)}{\mathfrak{c}^{(p+1)}} - \frac{\Psi(\mathfrak{c}^p h)}{\mathfrak{c}^p},\ \forall\, h \in \Theta,\ \epsilon > 0. \tag{4.9}$$

From (4.9) and (4.8), we acquire

$$\begin{aligned}\mathcal{N}\left(\frac{\Psi(\mathfrak{c}^\phi x)}{\mathfrak{c}^\phi} - \Psi(h), \sum_{p=0}^{\phi-1} \frac{\epsilon\sigma^p}{\mathfrak{c}^{(p+1)}}\right) &\geqslant \min\left\{\mathcal{N}\left(\frac{\Psi(\mathfrak{c}^{p+1}x)}{\mathfrak{c}^{p+1}} - \frac{\Psi(\mathfrak{c}^p h)}{\mathfrak{c}^i}, \frac{\epsilon\sigma^p}{\mathfrak{c}^{(p+1)}}\right)\right\} \\ &\geqslant \mathcal{N}'(\Gamma(h, h, \ldots, h), \epsilon),\ \forall\, h \in \Theta,\ \epsilon > 0.\end{aligned} \tag{4.10}$$

Replacing h by $\mathfrak{c}^m h$ in (4.10) and using (4.1), (N_3), we reach

$$\mathcal{N}\left(\frac{\Psi(\mathfrak{c}^{\phi+m}x)}{\mathfrak{c}^{(\phi+m)}} - \frac{\Psi(\mathfrak{c}^m h)}{\mathfrak{c}^m}, \sum_{p=0}^{\phi-1} \frac{\epsilon\sigma^p}{\mathfrak{c}^{(p+1)}}\right) \geqslant \mathcal{N}'(\Gamma(h, h, \ldots, h), \epsilon) \geqslant \mathcal{N}'(\Gamma(h, h, \ldots, h), \frac{\epsilon}{\sigma^m})$$

and so,

$$\mathcal{N}\left(\frac{\Psi(\mathfrak{c}^{\phi+m}x)}{\mathfrak{c}^{(\phi+m)}} - \frac{\Psi(\mathfrak{c}^m h)}{\mathfrak{c}^m}, \sum_{p=0}^{\phi-1} \frac{\epsilon\sigma^p}{\mathfrak{c}^{(p+1)}}\right) \geqslant \mathcal{N}'(\Gamma(h, h, \ldots, h), \epsilon),\ \forall\, h \in \Theta, \epsilon > 0,\ m, \phi \geqslant 0.$$

Replacing ϵ by $\frac{\epsilon}{\sum_{p=m}^{\phi+m-1} \frac{\sigma^p}{\mathfrak{c}^{(p+1)}}}$, we get

$$\mathcal{N}\left(\frac{\Psi(\mathfrak{c}^{\phi+m}x)}{\mathfrak{c}^{(\phi+m)}} - \frac{\Psi(\mathfrak{c}^m h)}{\mathfrak{c}^m}, \epsilon\right) \geqslant \mathcal{N}'\left(\Gamma(h, h, \ldots, h), \frac{\epsilon}{\sum_{p=m}^{\phi+m-1} \frac{\sigma^p}{(\mathfrak{c}^{(p+1)})}}\right),\ \forall\, h \in \Theta, \epsilon > 0,\ m, \phi \geqslant 0. \tag{4.11}$$

Since $\sum_{p=0}^{m}\left(\frac{\sigma}{c}\right)^{p} < \infty$ is a Cauchy criterion, which is convergence at $0 < \sigma < c$.

Hence, the sequence $\left\{\frac{\Psi(c^{\phi}h)}{c^{\phi}}\right\}$ is a Cauchy sequence in a fuzzy Banach space $(\mathcal{Y}, \mathcal{N})$. Also, this sequence converges to some point $A(x) \in \mathcal{Y}$. So we define a function $A : \Theta \to \mathcal{Y}$ by

$$A(h) = \mathcal{N} - \lim_{\phi\to\infty} \frac{\Psi(c^{\phi}h)}{c^{\phi}}, \ \forall\, h \in \Theta.$$

Note that, Ψ and A are odd functions. Letting $m = 0$ in equation (4.11), we obtain

$$\mathcal{N}\left(\frac{\Psi(c^{n}x)}{c^{\phi}} - \Psi(h), \epsilon\right) \geqslant \mathcal{N}'\left(\Gamma(h, h, \dots, h), \frac{\epsilon}{\sum_{p=m}^{\phi+m-1}\frac{\sigma^{p}}{(c^{(p+1)})}}\right), \quad \forall\, h \in \Theta, \epsilon > 0. \tag{4.12}$$

Taking the limit as $\phi \to \infty$ in (4.12) and using (N_6), we get

$$\mathcal{N}(\Psi(h) - A(h), \epsilon) \geqslant \mathcal{N}'(\Gamma(h, h, \dots, h), \epsilon(c - \sigma)), \ \forall\, h \in \Theta, \ \epsilon > 0.$$

Now, we claim that A is additive.

Varying $(h_1, h_2, \dots, h_{\phi})$ by $(c^{\phi}h_1, c^{\phi}h_2, \dots, c^{\phi}h_{\phi})$ in (4.2) respectively, we have

$$\mathcal{N}\left(\frac{1}{c^{\phi}} D\Psi(c^{\phi}h_1, c^{\phi}h_2, \dots, c^{\phi}h_{\phi}), \epsilon\right) \geqslant \mathcal{N}'(\Gamma(c^{\phi}h_1, c^{\phi}h_2, \dots, c^{\phi}h_{\phi}), c^{\phi}\epsilon), \ \forall\, h \in \Theta, \ \epsilon > 0.$$

Since $\lim_{\phi\to\infty} \mathcal{N}'(\Gamma(c^{\phi}h_1, c^{\phi}h_2, \dots, c^{\phi}h_{\phi}), c^{\phi}\epsilon) = 1$ and a function $A : \Theta \to \mathcal{Y}$ satisfies the functional equation (1.2). Hence, $A : \Theta \to \mathcal{Y}$ is an additive function.

To prove the uniqueness of $A : \Theta \to \mathcal{Y}$, let $B : \Theta \to \mathcal{Y}$ be another mapping satisfying (4.3). Fix $h \in \Theta$, clearly $A(c^{\phi}h) = c^{\phi}A(h)$ and $B(c^{\phi}h) = c^{\phi}B(h)$, $\forall\, h \in \Theta$, $\phi \in \mathcal{N}$. From (4.3), we have

$$\mathcal{N}(A(h) - B(h), \epsilon) = \mathcal{N}\left(\frac{A(c^{\phi}h)}{c^{\phi}} - \frac{B(c^{\phi}h)}{c^{\phi}}, \epsilon\right) \geqslant \mathcal{N}'\left(\Gamma(h, h, \dots, h), \frac{(c^{\phi})\epsilon(c - \sigma)}{2\sigma^{\phi}}\right), \ \forall\, h \in \Theta, \ \epsilon > 0.$$

Since $\lim_{\phi\to\infty} \frac{(c^{\phi})\epsilon(c-\sigma)}{2\sigma^{\phi}} = \infty$, we have

$$\lim_{\phi\to\infty} \mathcal{N}'\left(\Gamma(h, h, \dots, h), \frac{(c^{\phi})\epsilon(c - \sigma)}{2\sigma^{\phi}}\right) = 1.$$

Thus $\mathcal{N}(A(h) - B(h), \epsilon) = 1$, $\forall\, h \in \Theta$, $\epsilon > 0$ and so $A(h) = B(h)$. □

Corollary 4.2. *If a mapping* $\Psi : \Theta \to \mathcal{Y}$ *satisfies the inequality*

$$\mathcal{N}(D\Psi(h_1, h_2, \dots, h_{\phi}), \epsilon) \geqslant \begin{cases} \mathcal{N}'(\kappa, \epsilon), \\ \mathcal{N}'(\kappa \sum_{p=1}^{\phi} \|h_p\|^{\iota}, \epsilon), \\ \mathcal{N}'(\kappa \prod_{p=1}^{\phi} \|h_p\|^{\iota}, \epsilon), \\ \mathcal{N}'(\kappa(\sum_{p=1}^{\phi} \|h_p\|^{\phi\iota} + \prod_{p=1}^{\phi} \|h_p\|^{\iota}), \epsilon), \ \forall\, h_1, h_2, \dots, h_{\phi} \in \Theta, \ \epsilon, \kappa > 0, \end{cases}$$

then there exists a unique additive mapping $A : \Theta \to \mathcal{Y}$ *such that*

$$\mathcal{N}(\Psi(h) - A(h), \epsilon) \geqslant \begin{cases} \mathcal{N}'(\kappa, |c - 1|\epsilon), & \\ \mathcal{N}'(\kappa\|h\|^{\iota}, \phi|c - c^{\iota}|\epsilon), & \iota \neq 1, \\ \mathcal{N}'(\kappa\|h\|^{\phi\iota}, |c - c^{\phi\iota}|\epsilon), & \iota \neq \frac{1}{\phi}, \\ \mathcal{N}'(\kappa\|h\|^{\phi\iota}, (\phi + 1)|c - c^{\phi\iota}|\epsilon), & \iota \neq \frac{1}{\phi}, \ \forall\, h \in \Theta, \ \epsilon > 0, \end{cases}$$

where $c = \phi(\phi^2 - 2\phi + 1)$.

5. Stability of additive functional equation in random normed space-Direct method

In this section, the authors discuss Hyers-Ulam satbility of functional equations (1.3) in random normed space using direct method.

Theorem 5.1. *Let* $q = \pm 1$ *and* $\Psi : \Theta \to \mathcal{Y}$ *be an additive mapping for which there exists a function* $\tau : \Theta^{\phi} \to D^{+}$ *with the condition*

$$\lim_{\ell\to\infty} \mathbb{T}_{p=0}^{\infty}\left(\tau_{(3^{\ell+p}h_1,3^{\ell+p}h_2,\ldots,3^{\ell+p}h_{\phi})}\left(3^{\ell+p+1}qs\right)\right) = 1, \text{ and}$$
$$\lim_{\ell\to\infty} \mathbb{T}_{p=0}^{\infty}\left(\tau_{(3^{\ell+p}h_1,3^{\ell+p}h_2,\ldots,3^{\ell+p}h_{\phi})}\left(3^{\ell+p+1}qs\right)\right) = \lim_{\ell\to\infty}\left(\tau_{3^{\ell}h_1,3^{\ell}h_2,\ldots,3^{\ell}h_{\phi}}\left(3^{\ell p}s\right)\right),$$

such that

$$\Upsilon_{D\Psi(h_1,h_2,\ldots,h_{\phi})}(s) \geqslant \tau_{(h_1,h_2,\ldots,h_{\phi})}(s),\ \forall\, h_1,h_2,\ldots,h_{\phi}\in\Theta,\ s>0.$$

Then there exists a unique additive function $A : \Theta \to \mathcal{Y}$ *satisfying* (1.2) *and*

$$\Upsilon_{A(h)-\Psi(h)}(s) \geqslant \mathbb{T}_{p=0}^{\infty}\left(\tau_{(3^{(p+1)q}h,3^{(p+1)q}h,3^{(p+1)q}h,0,\ldots,0)}\left((\phi-2)3^{(1+p)q}s\right)\right),\ \forall\, h\in\Theta,\ s>0,$$

where $h \in \Theta$ *and* $A(h)$ *is defined by*

$$\Upsilon_{A(h)}(s) = \lim_{k\to\infty}\Upsilon_{\frac{\Psi(3^{\ell q}h)}{3^{\ell q}}}(s),\ \forall\, h\in\Theta,\ s>0.$$

Corollary 5.2. *If a mapping* $\Psi : \Theta \to \mathcal{Y}$ *satisfies the inequality*

$$\Upsilon_{Df(h_1,h_2,\ldots,h_{\phi})}(s) \geqslant \begin{cases} \tau_{\epsilon}(s), & \\ \tau_{\epsilon}\sum_{p=1}^{\phi}\|h_p\|^{\iota}(s), & \iota\neq 1,\\ \tau_{\epsilon}\left(\prod_{p=1}^{\phi}\|h_p\|^{\iota}+\sum_{p=1}^{\phi}\|h_p\|^{\phi\iota}\right)(s), & \iota\neq\frac{1}{\phi},\ \forall h_1,h_2,\ldots,h_{\phi}\in\Theta, s>0,\end{cases}$$

then there exists a unique additive function $A : \Theta \to \mathcal{Y}$ *such that*

$$\Upsilon_{\Psi(h)-A(h)}(s) \geqslant \begin{cases} \tau_{\frac{\epsilon}{(\phi-2)|2|}}(s), \\ \tau_{\frac{3\epsilon\|h\|^{\iota}}{(\phi-2)|3-3^{\phi}|}}(s), \\ \tau_{\frac{3\epsilon\|h\|^{s}}{(\phi-2)|3-3^{\phi\iota}|}}(s), \quad \forall\, h\in\Theta,\ s>0.\end{cases}$$

6. Fixed point method-Stability of (1.2)

In this part, we investigate the Hyers-Ulam stability of (1.2) in fuzzy normed space via fixed point technique. Define

$$\tau_p = \begin{cases} \mathfrak{c}, & \text{if } p=0,\\ \frac{1}{\mathfrak{c}}, & \text{if } p=1,\end{cases}$$

where $\mathfrak{c} = \phi(\phi^2-2\phi+1)$ and $\Omega = \{s/s : \Theta\to\mathcal{Y}, s(0)=0\}$.

Theorem 6.1. *Let* $\Psi : \Theta \to \mathcal{Y}$ *and* $\Gamma : \Theta^{\phi} \to \mathbb{Z}$ *be mappings with the condition*

$$\lim_{k\to\infty}\mathcal{N}'\left(\Gamma(\tau_{p^{\ell}}h_1,\tau_{p^{\ell}}h_2,\ldots,\tau_{p^{\ell}}h_{\phi}),\tau_{p^{\ell}}\epsilon\right) = 1, \forall h_1,h_2,\ldots,h_{\phi}\in\Theta \text{ and } \epsilon>0$$

and satisfy the inequality

$$\mathcal{N}(D\Psi(h_1,h_2,\ldots,h_\phi),\epsilon) \geqslant \mathcal{N}'(\Gamma(h_1,h_2,\ldots,h_\phi),\epsilon), \forall h_1,h_2,\ldots,h_\phi \in \Theta \text{ and } \epsilon>0.$$

If $L = L[i]$ *such that*

$$\mathcal{N}'\left(L\frac{1}{\tau_p}\beta(\tau_p h),\epsilon\right) = \mathcal{N}'(\beta(h),\epsilon),\ \forall\ h\in\Theta,\ \epsilon>0,$$

then there exists a unique additive mapping $A:\Theta\to\mathcal{Y}$ *satisfying* (1.2) *and*

$$\mathcal{N}(\Psi(h)-A(h),\epsilon) \geqslant \mathcal{N}'\left(\frac{L^{1-i}}{1-\pounds}\beta(h),\epsilon\right),\ \forall\ h\in\Theta,\ \epsilon>0.$$

Corollary 6.2. *If a function* $\Psi:\Theta\to\mathcal{Y}$ *satisfies the inequality*

$$\mathcal{N}(D\Psi(h_1,h_2,\ldots,h_\phi),\epsilon) \geqslant \begin{cases} \mathcal{N}'(\kappa,\epsilon), \\ \mathcal{N}'(\kappa\sum_{p=1}^{\phi}\|h_p\|^{\iota},\epsilon), \\ \mathcal{N}'(\kappa\prod_{p=1}^{\phi}\|h_p\|^{\iota},\epsilon), \\ \mathcal{N}'(\kappa(\sum_{p=1}^{\phi}\|h_p\|^{\phi\iota}+\prod_{p=1}^{\phi}\|h_p\|^{\iota}),\epsilon), & \kappa>0, \end{cases}$$

$\forall\ h_1,h_2,\ldots,h_\phi\in\Theta,\ \epsilon>0$, *then there exists a unique additive mapping* $A:\Theta\to\mathcal{Y}$ *such that*

$$\mathcal{N}(\Psi(h)-A(h),\epsilon) \geqslant \begin{cases} \mathcal{N}'(\kappa,|\mathfrak{c}-1|\epsilon), \\ \mathcal{N}'(\kappa\|h\|^{\iota},\phi|\mathfrak{c}-\mathfrak{c}^{\iota}|\epsilon), & \iota\neq 1, \\ \mathcal{N}'(\kappa\|h\|^{\phi\iota},|\mathfrak{c}-\mathfrak{c}^{\phi\iota}|\epsilon), & \iota\neq\frac{1}{\phi}, \\ \mathcal{N}'(\kappa\|h\|^{\phi\iota},(\phi+1)|\mathfrak{c}-\mathfrak{c}^{\phi\iota}|\epsilon), & \iota\neq\frac{1}{\phi},\ \ \forall\ h\in\Theta,\ \epsilon>0, \end{cases}$$

where $\mathfrak{c}=\phi(\phi^2-2\phi+1)$.

7. Stability of additive functional equation in random normed space-Fixed point method

In this section, the authors present the generalized Hyers-Ulam stability of functional equitation (1.3) in random normed space using fixed point method.

Theorem 7.1. *Let* $\Psi:\Theta\to\mathcal{Y}$ *and* $\tau:X^{\phi}\to D^{+}$ *be mappings with the condition*

$$\lim_{\ell\to\infty}\left((\tau_{\ell^p}h_1,\tau_{\ell^p}h_2,\ldots,\tau_{\ell^p}h_\phi),\tau_{\ell^p}s\right)=1,\forall\ h_1,h_2,\ldots,h_\phi\in\Theta,\ s>0,$$

where $\tau_p=\begin{cases}3, & p=0,\\ \frac{1}{3}, & p=1,\end{cases}$ *satisfies the functional inequality*

$$\Upsilon D\Psi(h_1,h_2,\ldots,h_\phi)(s)\geqslant\tau_{(h_1,h_2,\ldots,h_\phi)}(s),\ \ \forall\ h_1,h_2,\ldots,h_\phi\in\Theta,\ s>0.$$

If $\pounds=\pounds(i)$, *then* $h\to\beta(h,s)=\frac{1}{(n-2)}\tau_{(\frac{h}{3},\frac{h}{3},\frac{h}{3},0,\ldots,0)}(s)$ *has the property that*

$$\beta(h,s)\leqslant\pounds\frac{1}{\delta_p}\beta(\delta_p h,s),\ \ \forall h\in\Theta,s>0, \tag{7.1}$$

then there exists a unique additive mapping $A:\Theta\to\mathcal{Y}$ *satisfying* (1.2) *and*

$$\Upsilon_{A(h)-\Psi(h)}\left(\frac{L^{1-i}}{1-\pounds}t\right)\geqslant\beta(h,s),\ \ \forall h\in\Theta,s>0.$$

Proof. Let $d(h,g) = \inf\{\ell \in (0,\infty) / \Upsilon_(h(h) - g(h))(\ell s) \geqslant \beta(h,s), h \in \Theta, s > 0\}$. Then the normed space (Ω, d) is complete. Define $\mathcal{T} : \Omega \to \Omega$ by $\mathcal{T}h(h) = \frac{1}{\delta_p} h(\delta_p h)$, $\forall\, h \in \Theta$. Now

$$d(\mathcal{T}h, \mathcal{T}g) \leqslant \pounds d(h,g), \quad \forall\, h, g \in \Omega, \;\; d(h,g) \leqslant \ell.$$

Therefore $\mathcal{T}$ is strictly contractive mapping on Ω with Lipschitz constant £. Replacing $(h_1, h_2, \ldots, h_\phi)$ by $(h, h, h, 0, \ldots, 0)$ in (4.2), we get

$$\Upsilon_{(\phi-2)\Psi(3h)-3(n-2)\Psi(h)}(s) \geqslant \tau_{(h,h,h,0,\ldots,0)}(s), \quad \forall\, h \in \Theta.$$

From (4.6), we get

$$\Upsilon_{\frac{\Psi(3h)}{3}-\Psi(h)}(s) \geqslant \tau_{(h,h,h,0,\ldots,0)}((\phi-2)3s), \quad \forall\, h \in \Theta. \tag{7.2}$$

Using (7.1) for $\mathfrak{p} = 0$, it reduces to

$$\Upsilon_{\frac{\Psi(3h)}{3}-\Psi(h)}(s) \geqslant L\beta(h,s), \quad \forall\, h \in \Theta.$$

Hence,

$$d\left(\Upsilon_{\mathcal{T}\Psi(h)-\Psi(h)}\right) \geqslant L = L^{1-i} < \infty, \quad \forall\, h \in \Theta. \tag{7.3}$$

Replacing Θ by $\frac{x}{3}$ in (7.2), we get

$$\Upsilon_{\frac{\Psi(h)}{3}-\Psi\frac{h}{3}}(s) \geqslant \tau_{(\frac{h}{3},\frac{h}{3},\frac{h}{3},0,\ldots,0)}((\phi-2)3s), \quad \forall\, h \in \Theta.$$

Using (7.1) for $\mathfrak{p} = 1$, it reduces to

$$\Upsilon_{3\Psi\frac{h}{3}-\Psi(h)}(s) \geqslant \beta(h,s) \Rightarrow \Upsilon_{\mathcal{T}\Psi(h)-\Psi(h)}(s) \geqslant \beta(h,s), \quad \forall\, h \in \Theta.$$

Hence,

$$d\left(\Upsilon_{\mathcal{T}\Psi(h)-\Psi(h)}\right) \geqslant L = L^{1-i} < \infty, \quad \forall\, h \in \Theta. \tag{7.4}$$

From (7.3) and (7.4), we can conclude

$$d\left(\Upsilon_{\mathcal{T}\Psi(h)-\Psi(h)}\right) \geqslant L = L^{1-i} < \infty, \quad \forall\, h \in \Theta.$$

In order to prove that $A : \Theta \to \mathcal{Y}$ fulfills (1.2), the remaining proof is similar. Since A is a unique fixed point of $\mathcal{T}$ and $\triangle = \{\Psi \in \Omega / d(\Psi, A) < \infty\}$. Finally, A is an unique function such that

$$\Upsilon_{A(h)-\Psi(h)}\left(\frac{L^{1-i}}{1-\pounds}s\right) \geqslant \beta(h,s), \;\forall\;\; h \in \Theta, \;\; s > 0.$$

□

Corollary 7.2. *If a function $\Psi : \Theta \to \mathcal{Y}$ satisfies the inequality*

$$\Upsilon_{D\Psi(h_1,h_2,\ldots,h_\phi)}(s) \geqslant \begin{cases} \tau_\epsilon(s), & \\ \tau_\epsilon \sum_{\mathfrak{p}=1}^{\phi} \|h_\mathfrak{p}\|^\iota(s), & \iota \neq 1, \\ \tau_\epsilon \left(\prod_{\mathfrak{p}=1}^{\phi} \|h_\mathfrak{p}\|^\iota + \sum_{\mathfrak{p}=1}^{\phi} \|h_\mathfrak{p}\|^{\phi\iota}\right)(s), & \iota \neq \frac{1}{\phi}, \end{cases}$$

for all $h_1, h_2, \ldots, h_\phi \in \Theta$ and all $s > 0$, then there exists an additive function $A : \Theta \to \mathcal{Y}$ such that

$$\Upsilon_{\Psi(h)-A(h)}(s) \geqslant \begin{cases} \tau_{\frac{\epsilon}{(\phi-2)|2|}}(s), \\ \tau_{\frac{3\epsilon\|h\|^\iota}{(\phi-2)|3-3^\iota|}}(s), \\ \tau_{\frac{3\epsilon\|h\|^\iota}{(\phi-2)|3-3^{\phi\iota}|}}(s), \quad \forall\, h \in \Theta, \; s > 0. \end{cases}$$

8. Applications of functional equations

Many areas of mathematics use functional equations, including physics, geometry, measure theories, statistics, algebraic geometry, and group theory. Many intriguing applications of functional equations may be found in characterization issues in probability theory. Joint distributions formed from conditional distributions can be described using functional equation solutions. Stochastic processes, astronomy, economics, classical mechanics, computer graphics, dynamic programming, neural networks, statistics, coding theory, information theory, fuzzy set theory, artificial intelligence, decision theory, population ethics, and many other fields all use functional equations in some way.

We'll provide an example to show how functional equations may be used to solve geometry difficulties. Consider the rectangle with a g base and an h height. Let the rectangle's area be $f(h, g)$.

As illustrated in Figure 1, split the rectangle horizontally into two sub-rectangles with heights of h_1 and h_2 and the same base g.

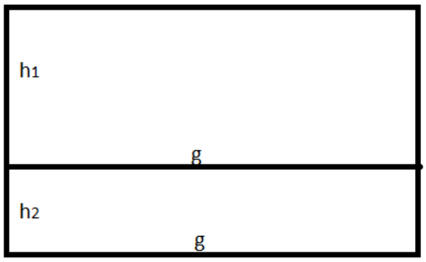

Figure 1: Divide the rectangle horizontally.

Then the area of sub-rectangles will be $f(h_1, g)$ and $f(h_2, g)$ and the area of the full rectangle is $f(h_1 + h_2, g)$. We have

$$f(h_1 + h_2, g) = f(h_1, g) + f(h_2, g). \tag{8.1}$$

In the same way as in Figure 2, we split the rectangle vertically with base heights $g1$ and $g2$ and the same height h.

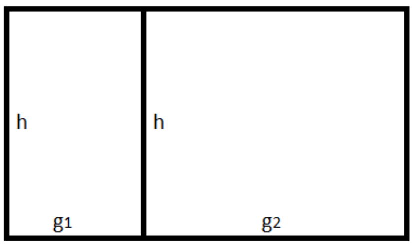

Figure 2: Divide the rectangle vertically.

Then the resulting areas are $f(h, g_1)$ and $f(h, g_2)$ and $f(h, g_1 + g_2)$. Therefore,

$$f(h, g_1 + g_2) = f(h, g_1) + f(h, g_2). \tag{8.2}$$

In equation (8.1), g is a constant and in equation (8.2), h is a constant. Both equations are similar to Cauchy's equation $f(x + y) = f(x) + f(y)$ whose solution is $f(x) = cx$. Therefore, the solution of (8.1) and (8.2) is

$$f(h, g) = c_1(g)h = c_2(h)g. \tag{8.3}$$

From (8.3),

$$\frac{c_1(g)}{g} = \frac{c_2(h)}{h} = c. \tag{8.4}$$

From (8.4),

$$c_1(g) = cg, c_2(h) = cg. \tag{8.5}$$

Substituting (8.5) in (8.3), we get

$$f(h, g) = chg,$$

where c is an arbitrary positive constant. Assume the initial conditions, that is, when $h = 1, g = 1$, the area of the rectangle = 1, which gives c = 1. Therefore, $f(h, g) = hg$. Hence, we arrive at the area of the rectangle.

9. Conclusion

We have introduced the general solution and generalized Hyers-Ulam stability of new type additive functional equations (1.2) and (1.3). Furthermore, we have proved the Hyers-Ulam stability for the additive functional equation (1.2) and (1.3) in fuzzy normed space and random normed space using two different methods. Also, we gave some applications of functional equations in physics. Through these examples, we explained how the functional equations appeared in the physical problem, and we talked about solutions that were not used for solving the problem, but which could be of interest. We provided an example to show how functional equations may be used to solve geometry difficulties.

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