



**Q-FRACTIONAL PROPORTIONAL DERIVATIVES
AND Q-LAPLACE TRANSFORMS**

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ABSTRACT

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DEMİR, TAYLAN
M.Sc. in Mathematics

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The main goal of this thesis is to develop a new type of q -fractional operators generated from proportional q -differences. To achieve this goal, first the main aspects and tools related to the q and q -fractional calculi are presented. After then, the proportional q -derivative is discussed. The proportional q -fractional differences or derivatives are proposed and the solutions of certain types of q -difference equations embodied by the proportional fractional derivatives are shown in details utilizing the q -Laplace transforms.

Keywords: q -proportional fractional derivative, q -fractional calculus, q - Laplace transform.

ÖZET

Q-KESİRLİ ORANTISAL TÜREVLER VE Q-LAPLACE DÖNÜŞÜMLERİ

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Matematik Yüksek Lisans

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Bu çalışmanın temel hedefi q -kesirli orantısal türevleri geliştirmektir. Bu hedefe ulaşmak için q ve q -kesirli analizleriyle ilgili bakış açıları ve araçları gösterilmiştir. Ondan sonra q -orantısal türevler tartışılmıştır. Q -kesirli orantısal türevler gösterildikten sonra bu operatörleri içeren bazı belirli denklemlerin çözümleri q -Laplace dönüşümü kullanılarak, detaylarıyla gösterilmiştir.

Anahtar Kelimeler: q -orantısal kesirli türev, q -kesirli kalkülüs, q -Laplace dönüşümü.

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LIST OF SYMBOLS

SYMBOLS

D	: Derivative Operator
I	: Integral Operator
D_q	: q-Derivative Operator
I_q	: q-Integral Operator
φ, ψ, Ω	: Functions
t	: Independent variable
c, d	: Constants
e_q	: q-analogues of the exponential functions
${}_q\mathcal{L}_s$: q-Laplace transformation
${}_qL_s$: q-Laplace transformation
$E_{\alpha,\theta}^\gamma(z)$: The generalized version of Mittag-Leffler function
$e_{\alpha,\theta}(z; q), E_{\alpha,\theta}(z; q)$: The q-analogues of the Mittag-Leffler functions
$B_q(y, x)$: q-beta function
$\Gamma_q(z)$: q-gamma function
$\ \varphi\ $: Norm of φ function
$\langle \varphi, \psi \rangle$: Inner product of φ and ψ functions
Γ	: Gamma Function
*	: Convolution
$\kappa_0(\beta, t)$: Constant of proportional operator
$\kappa_1(\beta, t)$: Constant of proportional operator
${}^cD_q^\theta$: q-Caputo fractional q-derivative of order θ
$\mathcal{AC}_q[0, b]$: The space of all q-absolutely continuous functions defined on $[0, b]$.

CHAPTER I

INTRODUCTION

In the last few decades, there has been a great deal of concern in what so called the fractional calculus which argues about the integrals and derivatives of non-integer order and thus it generalizes the classical calculus [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. This calculus turned out to be a good tool for modeling some real world problems. It was applied in physics by Hilfer [4] and was used in biomedical engineering by Magin [12]. In addition, many fractional mathematical models and their numerical techniques were given by Baleanu et al. [11] and Li et al. [13].

One of the best peculiarity of the fractional calculus is the fact that there are many types of the fractional operators. This enables a researcher to choose the most appropriate operator for the model under consideration. In spite of the diversity of the fractional operators, researchers have not stopped seeking for new fractional operators [14, 15, 16, 17]. There are also types of local fractional derivatives. One of these derivatives is the one called the conformable derivative proposed by Khalil et al. [18, 19, 20]. But the problem in this derivative is that it does not give the function itself when the order is 0. To overpass this problem, Anderson [21] proposed a modified version of the conformable derivative which give the function itself when the order is zero and the classical derivative when the order is one, and called it a proportional derivative. Jarad et al. [22, 23, 24, 25, 26, 27] launched out the nonlocal fractional versions of the proportional derivative.

The quantum calculus is a field of the derivative of the function that can be calculated without the assistance of the limit process and many mathematicians worked on q-calculus and their applications [28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41]. Moreover, the quantum calculus was applied with impulsive difference equations in [30], introduced with the operator theory in [33] and correlated with q-hypergeometric function in [35].

The fractional versions of the q -derivatives and integrals were discussed in details by [37, 42].

The aim of this master thesis is to introduce a new q -fractional operator called q -fractional proportional operator and to solve related q -fractional proportional differential equations with the help of q -Laplace transforms.

The overview of this thesis is as follows:

The proportional derivative is mentioned in Chapter 2. The concept of q -calculus is reviewed in Chapter 3. The q -proportional derivative and q -Laplace transform are provided in Chapter 4. The q -fractional proportional derivative and some related q -Laplace transforms are defined in Chapter 5. Chapter 6 is devoted to our conclusion.

CHAPTER II

PROPORTIONAL DERIVATIVE

The scope of this chapter is to provide the fundamentals and the main properties of the powerful fractional proportional derivative.

2.1 DEFINITION OF THE PROPORTIONAL DERIVATIVE

The proportional derivative is related with many fields of science and engineering and was used by many mathematicians [22, 27]. Anderson proposed the proportional derivative as a modification of the conformable derivatives [21]. It can also be defined with different time scales [43]. In fact, the proportional derivative is defined as

$$D^\beta \varphi(t) = \kappa_1(\beta, t)\varphi(t) + \kappa_0(\beta, t)\varphi'(t), \quad (2.1.1)$$

where $\beta \in [0,1]$ and $\kappa_0, \kappa_1: [0,1] \times \mathbb{R} \rightarrow [0, \infty)$ are continuous such that for all $t \in \mathbb{R}$,

$$\lim_{\beta \rightarrow 0^+} \kappa_1(\beta, t) = 1,$$

$$\lim_{\beta \rightarrow 0^+} \kappa_0(\beta, t) = 0,$$

$$\lim_{\beta \rightarrow 1^-} \kappa_1(\beta, t) = 0,$$

$$\lim_{\beta \rightarrow 1^-} \kappa_0(\beta, t) = 1$$

and $\kappa_1(\beta, t) \neq 0, \beta \in [0,1), \kappa_0(\beta, t) \neq 0, \beta \in (0,1]$. For a special interest, the case $\kappa_1(\beta, t) = 1 - \beta$ and $\kappa_0(\beta, t) = \beta$ will be considered. Therefore, (2.1.1) becomes (see[22]):

$$D^\beta \varphi(t) = (1 - \beta)\varphi(t) + \beta\varphi'(t). \quad (2.1.2)$$

2.2 SOME PROPERTIES OF PROPORTIONAL DERIVATIVE

Assume that the function $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $\kappa_0, \kappa_1: [0,1] \times \mathbb{R} \rightarrow [0, \infty)$ are continuous and satisfying the conditions in (2.1.1). Let φ and ψ be differentiable functions on \mathbb{R} . Then,

Property 1. $D^\beta [c\varphi + d\psi] = cD^\beta \varphi + dD^\beta \psi$ for all $c, d \in \mathbb{R}$;

Property 2. $D^\beta c = c\kappa_1(\beta, \cdot)$ for all $c \in \mathbb{R}$;

Property 3. $D^\beta (\varphi\psi) = \varphi D^\beta \psi + \psi D^\beta \varphi - \varphi\psi\kappa_1(\beta, \cdot)$;

Property 4. $D^\beta (\varphi/\psi) = \frac{\psi D^\beta \varphi - \varphi D^\beta \psi}{\psi^2} + \frac{\varphi}{\psi} \kappa_1(\beta, \cdot)$;

Property 5. For $\beta \in (0,1], m \in \mathbb{R}$, we have

$$D_t^\beta [e_n(t, m)] = n(t)e_n(t, m) \quad (2.2.1)$$

and

$$e_n(t, m) := e^{\int_m^t \frac{n(\tau) - \kappa_1(\beta, \tau)}{\kappa_0(\beta, \tau)} d\tau},$$

$$e_0(t, m) := e^{-\int_m^t \left(\frac{\kappa_1(\beta, \tau)}{\kappa_0(\beta, \tau)}\right) d\tau}.$$

Property 6. For $\beta \in (0,1]$, we have

$$D^\beta \left[\int_a^t \frac{\varphi(m)e_0(t, m)}{\kappa_0(\beta, m)} dm \right] = \varphi(t). \quad (2.2.2)$$

For a better understanding of this concept, the proofs are given in below.

Proof of Property 1[21]:

By direct calculations, one may conclude that

$$\begin{aligned} D^\beta [c\varphi + d\psi] &= \kappa_0 [c\varphi + d\psi]' + \kappa_1 \varphi \psi \\ &= c\kappa_0 \varphi' + \varphi \kappa_1 \psi + d\kappa_0 \psi' + d\kappa_1 \varphi + \kappa_1 \varphi \psi - \kappa_1 \varphi \psi \\ &= cD^\beta \varphi + dD^\beta \psi \end{aligned}$$

■

Proof of Property 2[21]:

By using the definition of the operator, it is obvious that

$$D^\beta c = c\kappa_1(\beta, \cdot), \quad c \in \mathbb{R}$$

■

Proof of Property 3[21]:

Operating D^β on $(\varphi\psi)$, one gets

$$\begin{aligned} D^\beta (\varphi\psi) &= \kappa_0 (\varphi\psi' + \varphi'\psi) + \kappa_1 \varphi \psi \\ &= (\varphi \kappa_0 \psi' + \varphi \kappa_1 \psi) + (\psi \kappa_0 \varphi' + \psi \kappa_1 \varphi) - \varphi \psi \kappa_1 \\ &= \varphi D^\beta \psi + \psi D^\beta \varphi - \varphi \psi \kappa_1. \end{aligned}$$

■

Proof of Property 4 [21]:

Applying the operator D^β on the ratio φ/ψ , one concludes that

$$\begin{aligned} D^\beta[\varphi/\psi] &= \frac{\kappa_0(\varphi'\psi - \varphi\psi')}{\psi^2} + \frac{\varphi}{\psi}\kappa_1(\beta, \cdot) \\ &= \frac{\varphi'\kappa_0\psi - \varphi\kappa_0\psi'}{\psi^2} + \frac{\varphi}{\psi}\kappa_1(\beta, \cdot) \\ &= \frac{\psi D^\beta\varphi - \varphi D^\beta\psi}{\psi^2} + \frac{\varphi}{\psi}\kappa_1(\beta, \cdot) \end{aligned}$$

■

Proof of Property 5 [21]: Starting with

$D_t^\beta e_n(t, m) = n(t)e_n(t, m)$, one gets

$$\begin{aligned} D_t^\beta e_n(t, m) &= \kappa_0(\beta, t) \left(\frac{n(t) - \kappa_1(\beta, t)}{\kappa_0(\beta, t)} \right) e_n(t, m) + \kappa_1(\beta, t)e_n(t, m) \\ &= (n(t) - \kappa_1(\beta, t))e_n(t, m) + \kappa_1(\beta, t)e_n(t, m) \\ &= n(t)e_n(t, m) - \kappa_1(\beta, t)e_n(t, m) + \kappa_1(\beta, t)e_n(t, m) \\ &= n(t)e_n(t, m). \end{aligned}$$

so, $D_t^\beta e_n(t, m) = n(t)e_n(t, m)$ is obtained.

■

Proof of Property 6 [21]: For $\beta \in (0, 1]$, one obtains the following

$D^\beta \left[\int_a^t \frac{\varphi(m)e_0(t, m)}{\kappa_0(\beta, m)} dm \right] = \varphi(t)$. Therefore,

$$\begin{aligned} D^\beta \left[\int_a^t \frac{\varphi(m)e_0(t, m)}{\kappa_0(\beta, m)} dm \right] &= \\ &= \kappa_0(\beta, t) \frac{d}{dt} \left(\int_a^t \frac{\varphi(m)e_0(t, m)}{\kappa_0(\beta, m)} dm \right) + \kappa_1(\beta, t) \int_a^t \frac{\varphi(m)e_0(t, m)}{\kappa_0(\beta, m)} dm \\ &= \kappa_0(\beta, t) \left(-\frac{\kappa_1(\beta, t)}{\kappa_0(\beta, t)} \int_a^t \frac{\varphi(m)e_0(t, m)}{\kappa_0(\beta, m)} dm + \frac{\varphi(t)e_0(t, t)}{\kappa_0(\beta, t)} \right) \\ &\quad + \kappa_1(\beta, t) \int_a^t \frac{\varphi(m)e_0(t, m)}{\kappa_0(\beta, m)} dm \end{aligned}$$

$$\begin{aligned} &= -\kappa_1(\beta, t) \int_a^t \frac{\varphi(m)e_0(t, m)}{\kappa_0(\beta, m)} dm + \varphi(t)e_0(t, t) + \kappa_1(\beta, t) \int_a^t \frac{\varphi(m)e_0(t, m)}{\kappa_0(\beta, m)} dm \\ &= \varphi(t)e_0(t, t) \\ &= \varphi(t) \end{aligned}$$

■

CHAPTER III

Q-CALCULUS

In this chapter, a brief review of q-calculus is presented.

3.1 STANDARD EXPRESSIONS OF Q-DERIVATIVE AND Q-INTEGRAL

The q-differential form of a function φ is

$$d_q\varphi(t) = \varphi(qt) - \varphi(t). \quad (3.1.1)$$

This is similar to the Nabla-difference operator given by $\nabla\varphi(t) = \varphi(t+1) - \varphi(t)$. Let $\sigma \in \mathbb{R}$ be fixed, and B be subset of \mathbb{C} . B is defined to be σ -geometric if $\sigma t \in B$ whenever $t \in B$, [37]. Also, in case B of \mathbb{C} is a σ -geometric, it must absorb all geometric sequences in the form $\{t\sigma^m\}_{m=0}^{\infty}$ (see [37]), $t \in B$. If real or complex valued function φ is given on a q -geometric set B , $|q| \neq 1$ [37], the q-derivative of the function $\varphi(t)$ is shown as [28, 29, 34, 37, 39]

$$D_q\varphi(t) = \frac{\varphi(qt) - \varphi(t)}{qt - t}$$

or

$$D_q\varphi(t) = \frac{\varphi(qt) - \varphi(t)}{(q-1)t}, \quad (0 < q < 1). \quad (3.1.2)$$

The same definition was presented in [40-41]. Particularly, (3.1.2) is called the Jackson q-difference operator, the Euler-Jackson q-difference operator or the Euler-Heine-Jackson q-difference operator. When $0 \in B$ and for $|q| < 1$, the q-derivative at zero is given as [37]

$$D_q\varphi(0) = \lim_{m \rightarrow \infty} \frac{\varphi(tq^m) - \varphi(0)}{tq^m}, \quad t \in B \setminus \{0\}, \quad (3.1.3)$$

where the limit is said to exist and it does not depend on t [37]. When it is assumed that $|q| > 1$, the q-derivative at zero is introduced by (see [37]);

$$D_q\varphi(0) = D_{q^{-1}}\varphi(0).$$

The following are some properties of the q-derivative [28, 29].

1) (Linearity Property)

$$D_q(c\varphi(t) + d\psi(t)) = cD_q\varphi(t) + dD_q\psi(t), \quad c, d \in \mathbb{R} \quad (3.1.4)$$

Proof:

$$\begin{aligned} D_q(c\varphi(t) + d\psi(t)) &= \frac{(c\varphi(qt) + d\psi(qt)) - (c\varphi(t) + d\psi(t))}{qt - t} \\ &= \frac{c\varphi(qt) + d\psi(qt) - c\varphi(t) - d\psi(t)}{qt - t} \\ &= c \left(\frac{\varphi(qt) - \varphi(t)}{qt - t} \right) + d \left(\frac{\psi(qt) - \psi(t)}{qt - t} \right) \end{aligned}$$

and ($0 < q < 1$)

$$= cD_q\varphi(t) + dD_q\psi(t)$$

■

2) Q-Derivative of a Product

$$D_q(\varphi(t)\psi(t)) = \varphi(qt)D_q\psi(t) + \psi(t)D_q\varphi(t). \quad (3.1.5)$$

Proof: (3.1.5) is proved as

$$\begin{aligned} D_q(\varphi(t)\psi(t)) &= \frac{\varphi(qt)\psi(qt) - \varphi(t)\psi(t)}{qt - t} \\ &= \frac{\varphi(qt)\psi(qt) - \varphi(qt)\psi(t) + \varphi(qt)\psi(t) - \varphi(t)\psi(t)}{qt - t} \\ &= \varphi(qt) \left(\frac{\psi(qt) - \psi(t)}{qt - t} \right) + \psi(t) \left(\frac{\varphi(qt) - \varphi(t)}{qt - t} \right) \\ &= \varphi(qt)D_q\psi(t) + \psi(t)D_q\varphi(t), \quad (0 < q < 1). \end{aligned}$$

■

3) Q-Derivative of a Quotient

$$D_q \left(\frac{\varphi(t)}{\psi(t)} \right) = \frac{\psi(t)D_q\varphi(t) - \varphi(t)D_q\psi(t)}{\psi(t)\psi(qt)}. \quad (3.1.6)$$

Proof: (3.1.6) is demonstrated as;

$$\begin{aligned} D_q \left(\frac{\varphi(t)}{\psi(t)} \right) &= \frac{\frac{\varphi(qt)}{\psi(qt)} - \frac{\varphi(t)}{\psi(t)}}{qt - t} \\ &= \frac{\varphi(qt)\psi(t) - \varphi(t)\psi(qt)}{\psi(qt)\psi(t)(qt - t)} \end{aligned}$$

$$\begin{aligned}
&= \frac{\varphi(qt)\psi(t) - \varphi(t)\psi(t) + \varphi(t)\psi(t) - \varphi(t)\psi(qt)}{\psi(qt)\psi(t)(qt - t)} \\
&= \frac{\psi(t) \left(\frac{\varphi(qt) - \varphi(t)}{qt - t} \right) - \varphi(t) \left(\frac{\psi(qt) - \psi(t)}{qt - t} \right)}{\psi(qt)\psi(t)} \\
&= \frac{\psi(t)D_q\varphi(t) - \varphi(t)D_q\psi(t)}{\psi(qt)\psi(t)}, \quad (0 < q < 1).
\end{aligned}$$

■

The above proofs were handled keeping in mind that the function of φ and ψ are continuous. Similar properties can be implemented to q -integral. In fact, the q -integral is introduced (see [30]) as

$\varphi: A_t \rightarrow \mathbb{R}$ and $A_t = \{tq^n: n \in \mathbb{N} \cup \{0\}\} \cup \{0\}$ and

$$I_q\varphi(t) = \int_0^t \varphi(s)d_qs = \sum_{m=0}^{\infty} t(1-q)q^m\varphi(tq^m), \quad (3.1.7)$$

when $c, d \in A_t$. So that,

$$\int_c^d \varphi(s)d_qs = I_q\varphi(d) - I_q\varphi(c), \quad (3.1.8)$$

$$\int_c^d \varphi(s)d_qs = (1-q) \sum_{m=0}^{\infty} q^m [d\varphi(dq^m) - c\varphi(cq^m)], \quad (3.1.9)$$

and $(0 < q < 1)$ in (3.1.8) and (3.1.9).

According to (3.1.8) and (3.1.9) (see [30]), the result is obtained.

4) (Linearity Property)

$$I_q(\varphi(t) + \psi(t)) = I_q\varphi(t) + I_q\psi(t). \quad (3.1.10)$$

Proof of Linearity Property: (3.1.10) is shown as;

$$\begin{aligned}
I_q(\varphi(t) + \psi(t)) &= \int_0^t (\varphi(s) + \psi(s))d_qs \\
&= \int_0^t \varphi(s)d_qs + \int_0^t \psi(s)d_qs \\
&= \sum_{m=0}^{\infty} t(1-q)q^m\varphi(tq^m) + \sum_{m=0}^{\infty} t(1-q)q^m\psi(tq^m) \\
&= I_q\varphi(t) + I_q\psi(t).
\end{aligned}$$

■

5) (Linearity Property)

$$I_q(\varphi(t) - \psi(t)) = I_q\varphi(t) - I_q\psi(t). \quad (3.1.11)$$

Proof of Linearity Property: From (3.1.11) it is evident that

$$\begin{aligned} I_q(\varphi(t) - \psi(t)) &= \int_0^t (\varphi(s) - \psi(s))d_qs \\ &= \int_0^t \varphi(s)d_qs - \int_0^t \psi(s)d_qs \\ &= \sum_{m=0}^{\infty} t(1-q)q^m\varphi(tq^m) - \sum_{m=0}^{\infty} t(1-q)q^m\psi(tq^m) \\ &= I_q\varphi(t) - I_q\psi(t). \end{aligned}$$

■

Definition 3.1.1: Let φ be a function defined on a q -geometric set B . φ is q -integrable on B if and only if $\int_0^m \varphi(t)d_qt$ exists for every $m \in B$, (see [37]).

Definition 3.1.2: If $\lim_{m \rightarrow \infty} \varphi(tq^m) = \varphi(0)$ for every $m \in B$ then φ that is described on a q -geometric set B , $0 \in B$ is q -regular at zero. In addition to this, when B is also q^{-1} -geometric, φ is q -regular at infinity when there exists a constant c , (see [37]) such that; $\lim_{m \rightarrow \infty} \varphi(tq^{-m}) = c$ for all $t \in B$. Now, $\varphi(0^+)$ and $\varphi(0^-)$ are shown as;

$$\varphi(0^+) = \lim_{\substack{t \rightarrow \infty \\ n > 0}} \varphi(nq^t), \quad \varphi(0^-) = \lim_{\substack{t \rightarrow \infty \\ n < 0}} \varphi(nq^t).$$

When $B \subseteq \mathbb{R}$ is q -geometric and φ is a q -regular at zero function (see [37]). As a result, q -regularity at zero plays the role of continuity in the classical sense in some settings (see [37]). But, continuity at zero implies q -regularity at zero. However, the converse cannot be true.

3.2 MORE PROPERTIES OF THE Q-INTEGRALS

The q -integration by parts is given as, (see [37]),

$$\int_0^b \psi(t)D_q\varphi(t)d_qt = (\varphi\psi)(b) - \lim_{m \rightarrow \infty} (\varphi\psi)(bq^m) - \int_0^b D_q\psi(t)\varphi(qt)d_qt, \quad (3.2.1)$$

where φ and ψ are q -regular at zero and limit could be changed by $(\varphi\psi)(0)$.

Theorem 3.2.1: Let φ be a q -regular at zero defined on a q -geometric set B involving zero. Then

$$\bar{\varphi}(z) = \int_a^z \varphi(t) d_q t, \quad (z \in B)$$

where, $\bar{\varphi}$ is q -regular at zero such that $D_q \bar{\varphi}(z) = \varphi(z)$ for all $z \in B$. In addition, if c and d are two different points in B , then (see [37]);

$$\int_c^d D_q \varphi(t) d_q t = \varphi(d) - \varphi(c). \quad (3.2.2)$$

Theorem 3.2.2: [37] Let φ be a function defined on $[c, d]$, $0 \leq c \leq d$. Suppose that there exists a number α , $0 \leq \alpha < 1$ such that $t^\alpha \varphi(t)$ is continuous on $[c, d]$. Let,

$$\bar{\varphi}(z) = \int_a^z \varphi(t) d_q t, \quad z \in [c, d],$$

where a is a fixed point in $[c, d]$, (see [37]). Then, $\bar{\varphi}(z)$ is a continuous function in $[c, d]$.

Proof of Theorem 3.2.2: The proof is shown as [37],

$\psi(z) = z^\alpha \varphi(z)$ for every $z \in [c, d]$. z_0 is fix and $z_0 \in [c, d]$. Consider that $z_0 \neq 0$. Later,

$$\begin{aligned} \bar{\varphi}(z) - \bar{\varphi}(z_0) &= (1 - q) \sum_{s=0}^{\infty} z q^s \varphi(z q^s) - (1 - q) \sum_{s=0}^{\infty} z_0 q^s \varphi(z_0 q^s) \\ &= (1 - q) z^{1-\alpha} \sum_{s=0}^{\infty} z q^{s(1-\alpha)} [\psi(z q^s) - \psi(z_0 q^s)] \\ &\quad + z_0^\alpha (z^{1-\alpha} - z_0^{1-\alpha}) (1 \\ &\quad - q) \sum_{s=0}^{\infty} \psi^s \psi(z_0 q^s) \end{aligned} \quad (3.2.3)$$

Because, $\psi(x)$ is continuous on $[c, d]$, $\psi(x)$ is uniformly continuous on $[c, d]$, (see [37]). Therefore, $\delta > 0$ exists for every $\epsilon > 0$ in fact for every $z, w \in [c, d]$ (see [37]),

$$|z - w| < \delta \rightarrow |\psi(z) - \psi(w)| < \epsilon.$$

So, when $z \in [c, d]$, (see [37]),

$|z - z_0| < \delta$ then $|z q^s - z_0 q^s| < \delta$ for every $s \in \mathbb{N}_0$, and in [37],

$|\psi(z q^s) - \psi(z_0 q^s)| < \epsilon$ for every $s \in \mathbb{N}_0$. Therefore, $\lim_{z \rightarrow z_0} \psi(z q^s) = \psi(z_0 q^s)$

uniformly in s and $\lim_{z \rightarrow z_0}$ can be applied on the series. Together with limit as $z \rightarrow z_0$

on the series could be introduced on (3.2.3) to satisfy $\lim_{z \rightarrow z_0} \bar{\varphi}(z) = \bar{\varphi}(z_0)$. Also, consider that $z_0 = 0$. So,

$$\bar{\varphi}(z) - \bar{\varphi}(0) = \int_0^z \varphi(t) d_q t = \int_0^z t^{1-\alpha} (\psi(t) - \psi(0)) + \frac{1-q}{1-q^{2-\alpha}} z^{2-\alpha} \psi(0).$$

As a result,

$$|\bar{\varphi}(z) - \bar{\varphi}(0)| \leq \left(\max_{s \in \mathbb{N}_0} |\psi(zq^s) - \psi(0)| + \psi(0) \right) \frac{1-q}{1-q^{2-\alpha}} z^{2-\alpha}.$$

Since, ψ is continuous at 0 and $0 < \alpha < 1$, $\lim_{z \rightarrow 0} \bar{\varphi}(z) = \bar{\varphi}(0)$ is obtained. So, $\bar{\varphi}(z)$ is continuous on $[c, d]$. ■

Lemma 3.2.1: In [37], let $u(t, s)$ be a function defined on $[0, b] \times [0, b]$ in fact for every fixed t the functions (see [37]),

$$D_{q,z}^j u(t, z) \quad (j = 0, 1, 2, \dots, s-1)$$

are q -integrable on $[0, b]$. When for some $z \in (0, b]$ and $s \in \mathbb{N}$ then

$$u(zq^v, zq^j) = 0, \quad (v = 0, 1, 2, \dots, j-1; j = 1, 2, \dots, s) \quad (3.2.4)$$

then

$$D_{q,z}^s \int_0^z u(t, z) d_q t = \int_0^z D_{q,z}^s u(t, z) d_q t.$$

Proof of Lemma 3.2.1: The m -th order q -derivative, D_q^m , of a function φ could be given as its values at the points $\{q^j z, j = 0, 1, \dots, m\}$ through the identity,

$$D_q^m \varphi(z) = (-1)^m (1-q)^{-m} z^{-m} q^{-m(m-1)/2} \sum_{v=0}^m (-1)^v \begin{bmatrix} m \\ v \end{bmatrix}_q q^{v(v-1)/2} \varphi(zq^{m-v})$$

for all z in $B\{0\}$. Then,

$$D_{q,z}^s \int_0^z u(t, z) d_q t = \sum_{j=0}^{j=s} (-1)^j \begin{bmatrix} s \\ j \end{bmatrix}_q \frac{q^{\frac{j(j+1)}{2} - sj}}{z^s (1-q)^s} \int_0^{zq^j} h(t, zq^j) d_q t, \quad (3.2.5)$$

and the means of φ obtains (3.2.4) implies

$$\int_0^{zq^j} u(t, zq^j) d_q t = \int_0^z u(t, zq^j) d_q t, \quad j = 1, 2, \dots, s.$$

Therefore,

$$\begin{aligned}
D_{q,z}^s \int_0^z u(t,z) d_q t &= \sum_{j=0}^{j=s} (-1)^j \begin{bmatrix} s \\ j \end{bmatrix}_q \frac{q^{\frac{j(j+1)}{2}-sj}}{z^s(1-q)^s} \int_0^z u(t, zq^j) d_q t \\
&= \int_0^z \left(\sum_{j=0}^{j=s} (-1)^j \begin{bmatrix} s \\ j \end{bmatrix}_q \frac{q^{\frac{j(j+1)}{2}-sj}}{z^s(1-q)^s} u(t, zq^j) \right) d_q t \\
&= \int_0^z D_{q,z}^s u(t,z) d_q t.
\end{aligned}$$

■

Lemma 3.2.2: [37] Assume that $u(t, z)$ is defined on $[b, \infty) \times [b, \infty)$ such that for every fixed t the functions,

$$D_{q,z}^j u(t, z), \quad (j = 0, 1, \dots, s-1)$$

are q -integrable on $[b, \infty)$. If for some x is defined on $[b, \infty)$ and $s \in \mathbb{N}$ $u(zq^v, zq^j) = 0$, $(v = 0, 1, 2, \dots, j-1; j = 1, 2, \dots, s)$, then

$$D_{q,z}^s \int_z^\infty u(t, z) d_q t = \int_z^\infty D_{q,z}^s u(t, z) d_q t.$$

Lemma 3.2.3: [37] Let I and J be intervals including zero such that $J \subseteq I$. Assume that φ_m, φ are functions defined on I , $m \in \mathbb{N}$, such that for every $t \in I$, φ_m tends uniformly to φ on J ; i.e.

$$\lim_{m \rightarrow \infty} \varphi_m(t) = \varphi(t), \quad (3.2.6)$$

Then,

$$\lim_{m \rightarrow \infty} \int_0^z \varphi_m(t) d_q t = \int_0^z \varphi(t) d_q t, \quad \forall z \in I, \quad (3.2.7).$$

3.3 IMPORTANT SPACES, FUNCTIONS AND NOTATIONS

3.3.1 Function Spaces

Let $1 \leq p < \infty$, $b > 0$ and $\xi \in \mathbb{R}$. If $L_{q,\xi}^p(0, b)$ is the space of all equivalence classes of functions having the property ([37]);

$$\int_0^b t^\xi |\varphi(t)|^p d_q t < \infty.$$

The norm defined on space $L_{q,\xi}^p(0, b)$ [37]

$$\|\varphi\|_{p,\xi,b} = \left(\int_0^b t^\xi |\varphi(t)|^p d_q t \right)^{1/p}$$

is a Banach space. In addition to this, when $p = 2$, the space $L_{q,\xi}^2(0,b)$ connected to the inner product

$$\langle \varphi, \psi \rangle = \int_0^b t^\xi \varphi(t) \bar{\psi}(t) d_q t, \quad (\varphi, \psi \in L_{q,\xi}^2(0,b)), \quad (3.3.1.1)$$

is a separable Hilbert space. In this case, an orthonormal basis of $L_{q,\xi}^2(0,b)$ is defined as

$$\varphi_n(x) = \begin{cases} \frac{1}{\sqrt{x^{\xi+1}(1-q)}}, & x = bq^n, \quad n \in \mathbb{N}_0, \\ 0, & \text{otherwise} \end{cases} \quad (3.3.1.2)$$

Especially, the case of $\xi = 0$ was shown in [44]. Also, for every $0 < q < 1$ and $0 < a < \infty$, the above mentioned Hilbert spaces are given by

$$L^p(\mathbb{R}_{a,q}) = \left\{ \varphi: \int_{-\infty/a}^{\infty/a} |\varphi(t)|^p d_q t < \infty \right\}, \quad (p \geq 1)$$

The notation \mathbb{R}_q and $\tilde{\mathbb{R}}_q$ should be used to demonstrate $\mathbb{R}_{1,q}$ and $\mathbb{R}_{\sqrt{1-q},q}$ in turn (see [37]), the inner product of $L^2(\mathbb{R}_{a,q})$ is

$$\langle \varphi, \psi \rangle = \int_{-\infty/a}^{\infty/a} \varphi(t) \bar{\psi}(t) d_q t, \quad (\varphi, \psi \in L^2(\mathbb{R}_{a,q}))$$

is a Hilbert space.

Definition 3.3.1.1: Let ξ be a real number and p be a positive number, then the space $\ell_{q,\xi}^p[0,b]$ is the space of all functions φ identified on $(0,b]$ obtaining

$$\|\varphi\|_{p,\xi} = \sup_{y \in (0,b]} \left(\int_0^y t^\xi |\varphi(t)|^p d_q t \right)^{1/p} < \infty. \quad (3.3.1.3)$$

The symbols $L_q^p[0,b]$, $\ell_q^p[0,b]$ and $\|\varphi\|_p$ are used to denote $L_{q,0}^p[0,b]$, $\ell_{q,0}^p[0,b]$ and $\|\varphi\|_{p,0}$ (see [37]).

Proposition 3.3.1.1: $(\ell_{q,\xi}^p[0, b], \|\cdot\|_{p,\xi})$ is a Banach space.

Proof of Proposition 3.3.1.1: It is enough to show that $(\ell_{q,\xi}^p[0, b], \|\cdot\|_{p,\xi})$ is a normed space directly. Consider $(\varphi_n)_n$ is a Cauchy sequence in $(\ell_{q,\xi}^p[0, b], \|\cdot\|_{p,\xi})$. Therefore, $n_0 \in \mathbb{N}$ exists for every $\epsilon > 0$ such that for every $n, m \in \mathbb{N}$.

$$n, m > n_0 \rightarrow \sup_{y \in (0, b]} \sum_{s=0}^{\infty} (xq^s)^{\xi+1} (1-q) |\varphi_n(xq^s) - \varphi_m(xq^s)|^p < \epsilon \quad (3.3.1.4)$$

so, $x^{\frac{\xi+1}{p}} \varphi_n(x)$ is a uniformly Cauchy sequence on $(0, b]$, (see [37]). Hence, φ which exists, identified on $(0, b]$ such that, [37];

$\lim_{n \rightarrow \infty} x^{\frac{\xi+1}{p}} \varphi_n(x) = x^{\frac{\xi+1}{p}} \varphi(x)$ is uniformly on $(0, b]$. Also, for the fix $M > 0$ and $n > n_0$, (3.3.1.4) is written as (see [37]);

$$m > n_0 \rightarrow \sum_{s=0}^M (xq^s)^{\xi+1} (1-q) |\varphi_n(xq^s) - \varphi_m(xq^s)|^p < \epsilon \quad \forall x \in (0, b], \quad (3.3.1.5)$$

Later limit is calculated as $m \rightarrow \infty$ on (3.3.1.5) gives for every $M > 0$ and $n > n_0$ (see [37]);

$$\sum_{s=0}^M (xq^s)^{\xi+1} (1-q) |\varphi_n(xq^s) - \varphi(xq^s)|^p \leq \epsilon, \quad \forall x \in [0, b].$$

Therefore, when $n \rightarrow \infty$ then $\|\varphi_n - \varphi\|_{p,\xi} \rightarrow 0$. As a result,

$$\varphi_{n_0+1} - \varphi \in (\ell_{q,\xi}^p[0, b], \|\cdot\|_{p,\xi})$$

and because $\varphi_{n_0+1} \in (\ell_{q,\xi}^p[0, b], \|\cdot\|_{p,\xi})$, then similarly $\varphi \in (\ell_{q,\xi}^p[0, b], \|\cdot\|_{p,\xi})$.

Definition 3.3.1.2: [37] Let $H_h(B)$ be the space of all functions determined on B when $\varphi \in H_h(B)$. Then there exists $c > 0$ such that;

$$|\varphi(x) - \varphi(0)| < c|x|^h, \quad \forall x \in B.$$

Definition 3.3.1.3: [37] $L_q^2((0, b) \times (0, b))$ is defined to be the space for every complex-valued functions $\varphi(x, t)$ defined on $[0, b] \times [0, b]$ such that,

$$\|\varphi(\cdot, \cdot)\|_2 = \left(\int_0^b \int_0^b |\varphi(x, t)|^2 d_q x d_q t \right)^{1/2} < \infty.$$

Lemma 3.3.1.1: [37] $L_q^2((0, b) \times (0, b))$ related with the inner product;

$$\langle \varphi, \psi \rangle_2 = \int_0^b \int_0^b \varphi(x, t) \overline{\psi(x, t)} d_q x d_q t$$

is a seperable Hilbert space.

Proof of Lemma 3.3.1.1: It is similar to the proof of lemma 3.3.1.1. It was noted in [44] that $L_q^2((0, b) \times (0, b))$ is a Banach space. It is enough now to show separability:

$$\Omega_{ij}(x, t) = \Omega_i(x)\Omega_j(t), \quad (i, j = 1, 2, \dots)$$

and Ω is an ortonormal basis of $L_q^2((0, b) \times (0, b))$ at anytime $\{\Omega_i(\cdot)\}_{i=1}^\infty$ is an orthonormal basis of $L_q^2(0, b)$. Actually,

$$\begin{aligned} \langle \Omega_{jk}, \Omega_{mn} \rangle_2 &= \int_0^b \int_0^b \Omega_j(x) \Omega_k(t) \overline{\Omega_m(x) \Omega_n(t)} d_q x d_q t \\ &= \int_0^b \Omega_j(x) \overline{\Omega_m(x)} d_q x \int_0^b \Omega_k(t) \overline{\Omega_n(t)} d_q t \\ &= w_{jm} w_{kn}, \end{aligned}$$

showing orthogonality. When $\varphi \in L_q^2((0, b) \times (0, b))$ exists $\{\Omega_{ij}\}$ can be proven to be a basis, such that $\langle \varphi, \Omega_{ij} \rangle_2 = 0$ for every $i, j \in \mathbb{N}_0$, then φ is the zero element (see [37]). Mainly,

$$\begin{aligned} 0 &= \langle \varphi, \Omega_{ij} \rangle = \int_0^b \int_0^b \varphi(x, t) \overline{\Omega_i(x) \Omega_j(t)} d_q x d_q t \\ &= \int_0^b \overline{\Omega_j(t)} \left(\int_0^b \varphi(x, t) \overline{\Omega_i(x)} d_q x \right) d_q t \\ &= \int_0^b u(t) \overline{\Omega_j(t)} d_q t. \end{aligned}$$

Hence,

$$u(t) = \int_0^b \varphi(x, t) \overline{\Omega_i(x)} d_q x$$

is orthogonal to the Ω_j 's implying that $\varphi(bq^n) = 0$, for every $n \in \mathbb{N}_0$, (see [37]). Therefore, from the above proof, $\varphi(x, bq^n)$ is orthogonal to each Ω_i , (see [37]). As a result, $\varphi(bq^m, bq^n) = 0$, for every $m, n \in \mathbb{N}_0$. ■

Definition 3.3.1.4: [37] $C_q^n[b, a]$ is defined as the space of all continuous functions that has continuous q -derivative up to order $n - 1$ on interval. Also, $C_q^n[b, a]$ related with the norm function

$$\|\varphi\| = \sum_{s=0}^{n-1} \max_{b \leq x \leq a} |D_q^s \varphi(t)|, \quad (\varphi \in C_q^n[b, a])$$

is a Banach space.

Lemma 3.3.1.2: $(C_q^n[b, a], \|\cdot\|)$, $n \in \mathbb{N}$, is a Banach space.

Proof of Lemma 3.3.1.2: In [37], let $(\varphi_m)_m$ be a Cauchy sequence in $C_q^n[b, a]$ and later, for every $\epsilon > 0$, $n_0 \in \mathbb{N}$ exists such that for every $k, m \in \mathbb{N}$,

$$k, m > n_0 \rightarrow \sum_{s=0}^{n-1} \max_{x \in [b, a]} |D_q^s \varphi_k(x) - D_q^s \varphi_m(x)| < \epsilon$$

Therefore,

$$k, m > n_0 \rightarrow \max_{x \in [b, a]} |D_q^s \varphi_k(x) - D_q^s \varphi_m(x)| < \epsilon.$$

In here, $(D_q^s \varphi_m)_m$ is a Cauchy sequence in $C[a, b]$ for $s = 0, 1, \dots, n - 1$. Hence,

$\psi_s \in C[b, a]$ exists for every $s \in \{0, 1, \dots, n - 1\}$. Then,

$$\lim_{s \rightarrow \infty} \max_{x \in [b, a]} |D_q^s \varphi(x) - \psi_s(x)| = 0, \quad s = 0, 1, \dots, n - 1.$$

then

$$\psi_k(x) = D_q^s \psi_0(x), \quad (x \in [b, a] \setminus \{0\}), \quad s = 0, 1, 2, \dots, n - 1 \quad (3.3.1.6).$$

when $0 \in (b, a)$ then,

$$\lim_{x \rightarrow 0} \psi_s(x) = \lim_{x \rightarrow 0} D_q^s \psi_0(x) = \lim_{v \rightarrow \infty} D_q^s \psi_0(tq^v),$$

for every $t \in (b, a)$ and $t \neq 0$. As a result,

$$\begin{aligned} \lim_{x \rightarrow 0} \psi_s(x) &= \lim_{v \rightarrow \infty} \frac{D_q^{s-1} \psi_0(tq^v) - D_q^{s-1} \psi_0(tq^{v+1})}{tq^v(1-q)} \\ &= \frac{1}{1-q} \lim_{v \rightarrow \infty} \left[\frac{D_q^{s-1} \psi_0(tq^v) - D_q^{s-1} \psi_0(0)}{tq^v} - q \frac{D_q^{s-1} \psi_0(tq^{v+1}) - D_q^{s-1} \psi_0(0)}{tq^{v+1}} \right] \\ &= D_q^s \psi_0(0) \end{aligned} \quad (3.3.1.7).$$

Therefore, the identity in (3.3.1.6) is valid for all $x \in [b, a]$ and so $\psi_0 \in C_q^n[b, a]$, (see [37]).

Definitin 3.3.1.5: [37]

$$C_\gamma[b, a] = \left\{ \psi(x): x^\gamma \psi(x) \in C[b, a], \|\psi\|_{C_\gamma} = \max_{b \leq x \leq a} |x^\gamma \psi(x)| \right\},$$

where $\gamma \in \mathbb{R}$.

3.3.2 Some q-functions

The q-shifted factorial function is presented for $\alpha \in \mathbb{C}$ by,

$$(z; q)_m = \begin{cases} 1, & m = 0 \\ \prod_{i=0}^{m-1} (1 - zq^i), & m \in \mathbb{N} \end{cases},$$

The limit of $(z; q)_m$ as $m \rightarrow \infty$ exists and is given by $(z; q)_\infty$. The multiple q-shifted factorial for complex numbers z_1, \dots, z_k is described as

$$(z_1, \dots, z_k; q)_m = \prod_{j=1}^k (z_j; q)_m.$$

Let α be a complex number. The following notation is used for the q-binomial coefficients,

$$\begin{bmatrix} \alpha \\ n \end{bmatrix}_q = \begin{cases} 1, & n = 0, \\ \frac{(1 - q^\alpha)(1 - q^{\alpha-1}) \dots (1 - q^{\alpha-n+1})}{(q; q)_n}, & n \in \mathbb{N}. \end{cases}$$

If $aq^\alpha \neq q^{-m}$ for all $m \in \mathbb{N}_0$, we define

$$(z; q)_\alpha = \frac{(z; q)_\infty}{(zq^\alpha; q)_\infty}$$

$$(-z; q)_\infty = \sum_{m=0}^{\infty} \frac{q^{m(m-1)/2}}{(q; q)_m} z^m \quad (z \in \mathbb{C}),$$

and

$$\frac{1}{(z; q)_\infty} = \sum_{m=0}^{\infty} \frac{z^m}{(q; q)_m}, \quad (|z| < 1)$$

were proposed by Euler. In [45, 46] these above expressions were mentioned. In [37], the above two expressions connect infinite products to infinite sums. In [47], $(-z; q)_\infty$ was pointed out as $E_q(z)$ and $1/(z; q)_\infty$ was explained by $e_q(z)$. Also, $(-z; q)_\infty$ and $1/(z; q)_\infty$ were defined as $E_q(z)$ and $e_q(z)$ in [45, 48, 49]. Therefore, $E_q(z)$ is an entire function with simple zeros at the points $\{-q^{-m}, m \in \mathbb{N}_0\}$, and

$$e_q(z)E_q(-z) \equiv 1, \quad |z| < 1 \tag{3.3.2.1}$$

So, the domain of the function $e_q(z)$ are able to be expanded to \mathbb{C} by defining $e_q(z)$, $z \in \mathbb{C}$, to be (see [37]),

$$e_q(z) = \frac{1}{(z; q)_\infty}.$$

and it relates with (3.3.2.1) and it holds in \mathbb{C} . Also, the function $e_q(z)$ has simple poles at the points $\{q^{-m}, m \in \mathbb{N}_0\}$, (see [47, 48]). In addition, to this, $e_q(z)$ was shown with series expansions in [48]. In [48], $e_q(z)$ was introduced also, that

$$e_q(z) = \frac{1}{(z; q)_\infty} = \sum_{m=0}^{\infty} \frac{q^{m^2-m}}{(q, z; q)_m} z^m$$

for $z \in \mathbb{C} \setminus \{q^{-s}, s \in \mathbb{N}_0\}$. Now, basic trigonometric functions $\sin_q z, \cos_q z, \text{Sin}_q z$ and $\text{Cos}_q z$ are defined by [37, 47].

$$\sin_q z = \frac{e_q(iz) - e_q(-iz)}{2i}, \quad |z| < 1, \quad (3.3.2.2)$$

$$\cos_q z = \frac{e_q(iz) + e_q(-iz)}{2}, \quad |z| < 1,$$

and

$$\text{Sin}_q z = \frac{E_q(iz) - E_q(-iz)}{2i}, \quad z \in \mathbb{C}, \quad (3.3.2.3)$$

$$\text{Cos}_q z = \frac{E_q(iz) + E_q(-iz)}{2}, \quad z \in \mathbb{C},$$

The functions $\sin_q z$ and $\cos_q z$ could be analytically continued through the identities (see [37]),

$$\sin_q z = \frac{\text{Sin}_q z}{(-z^2; q^2)_\infty}, \quad \cos_q z = \frac{\text{Cos}_q z}{(-z^2; q^2)_\infty}$$

for $z \in \mathbb{C} \setminus \{\pm q^{-m}i; m \in \mathbb{N}_0\}$.

In [37], $\sin_q z$ and $\cos_q z$ are meromorphic functions with poles at the points $\{\pm q^{-m}i, m \in \mathbb{N}_0\}$. At the same time, q -analogues of hyperbolic functions $\sinh z$ and $\cosh z$ are shown as;

$$\sinh_q z = -i \sin_q(iz), \quad \cosh_q z = \cos_q(iz), \quad (3.3.2.4)$$

$$\text{Sinh}_q z = -i \text{Sin}_q(iz), \quad \text{Cosh}_q z = \text{Cos}_q(iz). \quad (3.3.2.5)$$

When $z \in \mathbb{C} \setminus \{0\}, 0 < |q| < 1$ then α -function is determined by,

$$\alpha(z; q) = \sum_{m=-\infty}^{\infty} q^{m^2} z^m. \quad (3.3.2.6)$$

In [37], the following formula was introduced by Jacobi in 1829. And this identity is named by Jacobi's triple product identity (see [48]);

$$\sum_{m=-\infty}^{\infty} q^{m^2} z^m = (q^2; q^2)_{\infty} (-qz; q^2)_{\infty} (-qz^{-1}; q^2)_{\infty} \quad (3.3.2.7)$$

where $z \in \mathbb{C} \setminus \{0\}$ and $0 < |q| < 1$ (see [37, 48]). Therefore, $\alpha(z; q)$ has only real simple zeros at the points $\{-q^{2s+1}, s \in \mathbb{Z}\}$, (see [37]). A generalization of (3.3.2.7) is Ramanujan's identity;

$$\sum_{m=-\infty}^{\infty} \frac{(b; q)_m}{(a; q)_m} z^m = \frac{(bz, q/bz, q, a/b; q)_{\infty}}{(z, a/bz, a, q/b; q)_{\infty}}, \quad (3.3.2.8)$$

where $|q| < 1$ and $|ab^{-1}| < |z| < 1$, (see [37]).

3.3.3 The q-Gamma and q-Beta Functions

The q-Gamma is defined as by [50, 51]

$$\Gamma_q(z) = \frac{(q; q)_{\infty}}{(q^z; q)_{\infty}} (1 - q)^{1-z}, \quad (0 < |q| < 1) \quad (3.3.3.1)$$

where $z \in \mathbb{C} \setminus \{-m: m \in \mathbb{N}_0\}$ and the principal values of q^z and $(1 - q)^{1-z}$ is used. $\Gamma_q(z)$ is a meromorphic function with poles at $z = -m, m \in \mathbb{N}_0$, [37]. Because, $\Gamma_q(z)$ has no zeros, $1/\Gamma_q(z)$ is an entire function with zeros at $z = -m, m \in \mathbb{N}_0$, [37]). It can also be defined as,

$$\Gamma_q(m) = \frac{(q; q)_{m-1}}{(1 - q)^{m-1}}, \quad (m \in \mathbb{N}).$$

$\Gamma_q(y)$ satisfies the following property when $y > 0$.

$$\Gamma_q(y + 1) = \frac{1 - q^y}{1 - q} \Gamma_q(y), \quad \Gamma_q(1) = 1.$$

Now, the q-beta function is proposed as [37]

$$B_q(y, x) = \int_0^1 t^{y-1} (qt; q)_{x-1} d_q t, \quad (Re(y) > 0; Re(x) > 0) \quad (3.3.3.2)$$

The relation between q-beta of q-gamma functions was given in [52] as

$$B_q(y, x) = \frac{\Gamma_q(y)\Gamma_q(x)}{\Gamma_q(y+x)}, \quad (Re(y) > 0, \quad Re(x) > 0). \quad (3.3.3.3)$$

For more details on the properties of the q-gamma and the q-beta functions we refer the reader to [53, 54, 55, 56, 57].

Lemma 3.3.3.1: Assume that α and θ are two complex numbers. Then for all $m \in \mathbb{N}_0$ then,

$$(q^{\alpha+\theta}; q)_m = \sum_{s=0}^m \begin{bmatrix} m \\ s \end{bmatrix}_q q^{s\theta} (q^\alpha; q)_s (q^\theta; q)_{m-s} \quad (3.3.3.4)$$

Proof of Lemma 3.3.3.1: We have

$$\begin{aligned} & \sum_{s=0}^m \begin{bmatrix} m \\ s \end{bmatrix}_q q^{s\theta} (q^\alpha; q)_s (q^\theta; q)_{m-s} \\ &= (q^\theta; q)_m {}_2\phi_1(q^{-m}, q^\alpha; q^{1-\theta-m}; q, q) \\ &= (q^\theta; q)_m q^{m\alpha} \frac{(q^{1-\theta-\alpha-m}; q)_m}{(q^{1-\theta-m}; q)_m} = (q^{\alpha+\theta}; q)_m. \end{aligned}$$

Corollary 3.3.3.1: Assume that α and θ are complex numbers, and let $m \in \mathbb{N}_0$, (see [37]). Then,

$$B_q(\alpha, \theta)(q^{m+1}; q)_{\alpha+\theta-1} = \sum_{s=0}^m q^{s\theta} (1-q)(q^{s+1}; q)_{\alpha-1} (q^{m-s+1}; q)_{\theta-1} \quad (3.3.3.5)$$

and

$$\begin{aligned} & q^{-m\theta} (q^{m+1}; q)_{\alpha+\theta-1} B_q(\alpha, \theta) \\ &= \sum_{s=0}^m q^{-s\theta} (1-q)(q^{m-s+1}; q)_{\alpha-1} (q^{s+1}; q)_{\theta-1}. \end{aligned} \quad (3.3.3.6)$$

Proof of Corollary 3.3.3.1: This proof is done as (see [37]);

$$\begin{aligned} (q^{\alpha+\theta}; q)_m &= \sum_{s=0}^m \begin{bmatrix} m \\ s \end{bmatrix}_q q^{s\theta} (q^\alpha; q)_s (q^\theta; q)_{m-s} \\ \Gamma_q(z) &= \frac{(q; q)_\infty}{(q^z; q)_\infty} (1-q)^{1-z}, \quad (0 < |q| < 1) \end{aligned}$$

are used and one can easily see that for any $\alpha \in \mathbb{C}, j \in \mathbb{N}_0$;

$$(q^{j+1}; q)_{\alpha-1} = \frac{(q; q)_\infty (q^\alpha; q)_j}{(q^\alpha; q)_\infty (q; q)_j} = \Gamma_q(\alpha) (1-q)^{\alpha-1} \frac{(q^\alpha; q)_j}{(q; q)_j} \quad (3.3.3.7)$$

Therefore, (3.3.3.4) becomes

$$\begin{aligned} & \sum_{s=0}^m q^{s\theta} (1-q)(q^{s+1}; q)_{\alpha-1} (q^{m-s+1}; q)_{\theta-1} \\ &= \frac{(1-q)^{\alpha+\theta-1} \Gamma_q(\alpha) \Gamma_q(\theta)}{(q; q)_m} \sum_{s=0}^m q^{s\theta} \begin{bmatrix} m \\ s \end{bmatrix}_q (q^\alpha; q)_s (q^\theta; q)_{m-s} \end{aligned} \quad (3.3.3.8)$$

$$= (1-q)^{\alpha+\theta-1} \Gamma_q(\alpha) \Gamma_q(\theta) \frac{(q^{\alpha+\theta}; q)_m}{(q; q)_m}.$$

From (3.3.3.7), one can conclude that

$$\frac{(q^{\alpha+\theta}; q)_m}{(q; q)_m} = \frac{(1-q)^{1-\alpha-\theta}}{\Gamma_q(\alpha+\theta)} (q^{m+1}; q)_{\alpha+\theta-1} \quad (3.3.3.9)$$

■

Lemma 3.3.3.2: [37] Let α and θ be two complex numbers with positive real parts, and let $\varphi \in L_q^1(0, a)$ for some $a > 0$. Then,

$$\begin{aligned} a^{\alpha-1} \int_0^a (qt/a; q)_{\alpha-1} t^{\theta-1} \int_0^t (qv/t; q)_{\theta-1} \varphi(u) d_q u d_q t &= \\ &= B_q(\alpha, \theta) a^{\alpha+\theta-1} \int_0^a (qt/a; q)_{\alpha+\theta-1} \varphi(t) d_q t \end{aligned} \quad (3.3.3.10)$$

Proof of Lemma 3.3.3.2: Assume that $\alpha, \theta > 0$ and $\varphi \in L_q^1(0, a)$, $a > 0$. From the definition of the q -integration, one has

$$\begin{aligned} &a^{\alpha-1} \int_0^a (qt/a; q)_{\alpha-1} t^{\theta-1} \int_0^t (qv/t; q)_{\theta-1} \varphi(v) d_q v d_q t \\ &= a^{\alpha+\theta} (1-q)^2 \sum_{m=0}^{\infty} q^{m\theta} (q^{m+1}; q)_{\alpha-1} \sum_{n=0}^{\infty} q^{m+n} (q^{m+1}; q)_{\theta-1} \varphi(aq^{m+n}) \\ &= a^{\alpha+\theta} (1-q)^2 \sum_{m=0}^{\infty} q^{m\theta} (q^{m+1}; q)_{\alpha-1} \sum_{s=m}^{\infty} q^s (q^{s-m+1}; q)_{\theta-1} \varphi(aq^s) \\ &= a^{\alpha+\theta-1} \sum_{s=0}^{\infty} aq^s (1-q) \varphi(aq^s) \sum_{m=0}^s q^{m\theta} (1-q) (q^{m+1}; q)_{\alpha-1} (q^{s-m+1}; q)_{\theta-1}. \end{aligned}$$

Using (3.3.3.5), one obtains

$$\begin{aligned} &a^{\alpha-1} \int_0^a (qt/a; q)_{\alpha-1} t^{\theta-1} \int_0^t (qv/t; q)_{\theta-1} \varphi(v) d_q v d_q t \\ &= a^{\alpha+\theta-1} B_q(\alpha, \theta) \sum_{s=0}^{\infty} aq^s (1-q) (q^{s+1}; q)_{\alpha+\theta-1} \varphi(aq^s) \\ &= a^{\alpha+\theta-1} B_q(\alpha, \theta) \int_0^a (qt/a; q)_{\alpha+\theta-1} \varphi(t) d_q t. \end{aligned}$$

■

3.3.4 The q-Mittag Leffler Functions

The classical Mittag-Leffler functions are useful functions in the fractional calculus. In this chapter, the q-analogues of the Mittag-Leffler function are introduced. We are interested in a pair of q-Mittag-Leffler function which might be conceived as a generalization of the q-exponential functions $e_q(z)$ and $E_q(z)$.

First of all, the Mittag-Leffler functions are defined as

$$E_\gamma(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(m\gamma + 1)}, \quad (\gamma > 0), \quad (3.3.4.1)$$

and

$$E_{\gamma,\theta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(m\gamma + \theta)} \quad (\gamma > 0; \theta \in \mathbb{C}, z \in \mathbb{C}). \quad (3.3.4.2)$$

(3.3.4.1) is a one-parameter of Mittag-Leffler function, (3.3.4.2) is a two-parameter of Mittag-Leffler function. Especially, the generalized version of Mittag-Leffler function is introduced as

$$E_{\alpha,\theta}^\gamma(z) = \sum_{m=0}^{\infty} \frac{z^m (\gamma)_m}{m! \Gamma(m\alpha + \theta)}.$$

The q-analogues of the Mittag-Leffler functions have two main q-exponential functions, namely $e_q(z)$ and $E_q(z)$.

$$\begin{aligned} e_{\alpha,\theta}(z; q) &= \sum_{m=0}^{\infty} \frac{z^m}{\Gamma_q(m\alpha + \theta)}, \quad (|z(1-q)^\alpha| < 1), \\ E_{\alpha,\theta}(z; q) &= \sum_{m=0}^{\infty} \frac{q^{\alpha m(m-1)/2}}{\Gamma_q(m\alpha + \theta)} z^m, \quad z \in \mathbb{C}, \end{aligned} \quad (3.3.4.3)$$

where $\alpha > 0$, $\theta \in \mathbb{C}$. For $\theta = 1$, the functions $e_{\alpha,1}(z; q)$ and $E_{\alpha,1}(z; q)$ shows classes of q-exponential functions of one parameter, (see [37]). Other one parameter class of q-exponential functions is shown by [49, 58, 59],

$$E_q^{(\alpha)}(z) = \sum_{s=0}^{\infty} \frac{q^{as^2/2}}{(q; q)_s} z^s, \quad (\alpha \in \mathbb{C}).$$

In [60], other pair of q-Mittag-Leffler functions were defined $e_{q;\alpha,\theta}(z; c)$ and $E_{q;\alpha,\theta}(z; c)$ such that

$$e_{q;\alpha,\theta}(z; c) = \sum_{m=0}^{\infty} \frac{(c/z; q)_{\alpha m + \theta - 1}}{(q; q)_{\alpha m + \theta - 1}} z^{\alpha m + \theta - 1}, \quad (|z| > |c|),$$

$$E_{q;\alpha,\theta}(z; c) = \sum_{m=0}^{\infty} \frac{q^{\binom{\alpha m + \theta - 1}{2}} (c/z; q)_{\alpha m + \theta - 1}}{(q; q)_{\alpha m + \theta - 1} (-c; q)_{\alpha m + \theta - 1}}, \quad (z \in \mathbb{C}),$$

where $\{q, z, c, \alpha\} \subset \mathbb{C}$, $Re(\alpha), Re(\theta) > 0$ and $|q| < 1$. $e_{q;\alpha,\theta}(z; c)$ was called the small q-Mittag-Leffler function and $E_{q;\alpha,\theta}(z; c)$ was called as the big q-Mittag-Leffler function. Clearly, [37],

$$e_{q;\alpha,\theta}(z; 0) = (1 - q)^{-\theta} z^{\theta-1} e_{\alpha,\theta}(z^{\alpha}(1 - q)^{-\alpha}; q).$$

3.3.5 q-Analogues of the Laplace Transform

q-analogues of the Laplace transform was given in [61]. In fact, there are q-versions of the Laplace transform

$${}_qL_s \{\varphi(t)\} = \Omega(s) = \frac{1}{1 - q} \int_0^{s^{-1}} E_q(-qst) \varphi(t) d_q t, \quad (3.3.5.1)$$

and

$${}_q\mathcal{L}_s \{\varphi(t)\} = \mathbf{\Omega}(s) = \frac{1}{1 - q} \int_0^{\infty} e_q(-st) \varphi(t) d_q t, \quad (3.3.5.2)$$

where $Re(s) > 0$, (see [37]).

Properties of the ${}_qL_s$ Transform:

The most significant features of the ${}_qL_s$ transform needed in the sequel are summarized here. By using the definition of below the q-integration;

$$\int_0^t \varphi(\tau) d_q \tau = (1 - q) \sum_{m=0}^{\infty} t q^m \varphi(t q^m), \quad (t \in A), \quad (3.3.5.3)$$

the q-Laplace transform of (3.3.5.1) can be written as

$${}_qL_s \{\varphi(t)\} = \frac{(q; q)_{\infty}}{s} \sum_{j=0}^{\infty} \frac{q^j}{(q; q)_j} \varphi(s^{-1} q^j).$$

The convolution of two functions $\bar{\varphi}$ and $\bar{\psi}$ is defined [61] as

$$(\bar{\varphi} * \bar{\psi})(t) = \frac{t}{1 - q} \int_0^1 \bar{\varphi}(t\tau) \bar{\psi}(t - tq\tau) d_q \tau \quad (3.3.5.4)$$

where $\bar{\psi}[t - tq\tau]$, for (see [37]);

$$\bar{\psi}(t) = \sum_{m=0}^{\infty} a_m t^m,$$

is shown as,

$$\bar{\psi}[t - t_i] = \sum_{m=0}^{\infty} a_m [t - t_i]_m$$

with

$$[t - t_i]_m = t^m \left(\frac{t_i}{t}; q \right)_m$$

By using the definition of q -integration, then $(\bar{\varphi} * \bar{\psi})$ satisfies;

$$(\bar{\varphi} * \bar{\psi})(t) = \frac{1}{1-q} \int_0^t \bar{\varphi}(\tau) \bar{\psi}(t - q\tau) d_q \tau, \quad (3.3.5.5)$$

and thus $\bar{\psi}(t - q\tau) = \varepsilon^{-q\tau} \bar{\psi}(t)$ is defined in this manner. In [49], the convolution of two functions $\bar{\varphi}, \bar{\psi}$ is shown as

$$(\bar{\varphi} * \bar{\psi}) = \frac{1}{1-q} \int_0^t \bar{\varphi}(\tau) \varepsilon^{-q\tau} \bar{\psi}(t) d_q \tau, \quad (3.3.5.6).$$

In [37], it is remarked by Hahn (see [61]) that the convolution theorem

$${}_q L_s \{ \bar{\varphi} * \bar{\psi} \} = {}_q L_s \bar{\varphi} {}_q L_s \bar{\psi}, \quad (3.3.5.7)$$

valid only for ${}_q L_s$ transform and does not hold for the ${}_q L_s$ transform, (see [37]).

Now, there are some basic properties of the ${}_q L_s$ transform are mentioned. Initially, let

$${}_q L_s \{ \varphi(t) \} = \Omega(s).$$

Property 1:

$${}_q L_s \{ \varphi(bt) \} = (1/b) \Omega(s/b), \quad (b \neq 0).$$

Property 2:

$${}_q L_s \{ D_q^m \varphi(t) \} = \left(\frac{s}{1-q} \right)^m \Omega(s) - \sum_{k=1}^m D_q^{m-k} \varphi(0) \frac{s^{k-1}}{(1-q)^k}, \quad (m \in \mathbb{N}). \quad (3.3.5.8)$$

When the case $m = 1$ then, (see [37]),

$${}_q L_s \{ D_q \varphi(t) \} = \left(\frac{s}{1-q} \right) \Omega(s) - \frac{\varphi(0)}{1-q} = \frac{s\Omega(s) - \varphi(0)}{1-q}.$$

Property 3:

$${}_q L_s \left\{ \frac{\varphi(t)}{t} \right\} = \frac{1}{1-q} \int_{qs}^{\infty} \Omega(\tau) d_q \tau$$

and ${}_q L_s \left\{ \frac{\varphi(t)}{t} \right\}$ exists.

Property 4:

$${}_qL_s \left\{ \int_t^\infty \frac{\varphi(\tau)}{\tau} d_q \tau \right\} = \frac{1}{s} \int_0^{qs} \Omega(\tau) d_q \tau,$$

provided that [37]

- i) the two integral exists,
- ii) $\Omega(s)$ exists for all s ,
- iii) $q^j \sum_{r=1}^\infty |\varphi(q^j/s)| = O(|h|^j)$, for all fixed j and $|h| < 1$.

Property 5:

If

$$I_q^m \varphi(t) = \int_0^t \int_0^{t_{m-1}} \int_0^{t_{m-2}} \dots \int_0^{t_1} \varphi(\tau) d_q \tau d_q t_1 d_q t_2 \dots d_q t_{m-1}, \quad (3.3.5.9)$$

then

$${}_qL_s \{I_q^m \varphi(t)\} = \left(\frac{1-q}{s}\right)^m \Omega(s).$$

Property 6:

If ${}_qL_s \{\varphi_k(t)\} = \Omega_k(s)$ then

$${}_qL_s \left\{ \sum_{k=0}^m \varphi_k(t) \right\} = \sum_{k=0}^m \Omega_k(s) \quad (3.3.5.10)$$

holds if the following conditions hold

- a) m is finite
- b) m is infinite and
- i) $\sum_{k=0}^\infty |\varphi_k(tq^j)|$ is convergent for every $t_0 q^j$, t_0 being fixed,
- ii) $q^j \sum_{k=0}^\infty |\varphi_k(tq^j)| = O(h^j)$, where j is greater than some fixed J and h is a fixed quantity, $|h|$ being less than unity.

Property 7:

Set

$$\bar{\varphi}(t) = \int_0^t \frac{\varphi(\tau)}{\tau} d_q \tau$$

and consider that $\lim_{t \rightarrow 0} \frac{\varphi(t)}{t} = 0$. Then,

$${}_qL_s \{\bar{\varphi}(t)\} = \frac{1}{s} \int_{qs}^\infty \Omega(s) d_q s.$$

The inversion formula of the ${}_qL_s$ transform is given below and it was proposed in [61].

Definition of Hahn's inversion formula: If ${}_qL_s\{\varphi(t)\} = \Omega(s)$ then,

$$\varphi(t) = \int_c \Omega(s)e_q(st)ds, \quad (3.3.5.11).$$

where the path of integration C encircles the origin and could be deformed into a loop, parallel to the imaginary axis. Especially, if $\Omega(s)$ is analytic then according to the above, the inversion can be written as

$$\varphi(t) = \frac{1}{t} \sum_{i=0}^{\infty} (-1)^i q^{i(i-1)/2} \frac{\Omega(t^{-1}q^{-i})}{(q; q)_i}.$$

Definition of the ${}_q\mathcal{L}_s$ Transform: Now, even though there is no q -analogue of the convolution theorem existing for the ${}_q\mathcal{L}_s$ transform, the ${}_q\mathcal{L}_s$ transform has an advantage over the ${}_qL_s$ transform. This advantage is the recognition of q -analogue of the Goldstein theorem. By using the definition of below q -integration,

$$\int_0^t \varphi(\tau)d_q\tau = (1-q) \sum_{m=0}^{\infty} tq^m \varphi(tq^m), \quad (t \in A)$$

then the q -Laplace transform of (3.3.5.2) is,

$${}_q\mathcal{L}_s\varphi(t) = \frac{1}{(-s; q)_{\infty}} \sum_{j=-\infty}^{\infty} q^j (-s; q)_j \varphi(q^j). \quad (3.3.5.12)$$

Now, there are many basic properties of the ${}_q\mathcal{L}_s$ transform are introduced. Particularly, Abdi introduced the following properties of ${}_q\mathcal{L}_s$ transform [62].

Property 8: If ${}_q\mathcal{L}_s\{\varphi(t)\} = \Omega(s)$ and $b \in \mathbb{R}_{q,+}$ then,

$${}_q\mathcal{L}_s\{\varphi(bt)\} = (1/b)\Omega(s/b).$$

Property 9:

$${}_q\mathcal{L}_s\{D_q^m \varphi(t)\} = \left(\frac{s}{1-q}\right)^m \Omega(s) - \sum_{k=1}^m D_q^{m-k} \varphi(0) \frac{s^{k-1}}{q^{k-1}(1-q)^k}, \quad (3.3.5.13)$$

and $m \in \mathbb{N}$ in (3.3.5.13).

Property 10: In [37],

$${}_q\mathcal{L}_s\{\varphi(t)/t\} = \frac{1}{1-q} \int_{qs}^{\infty} \Omega(\tau) d_q\tau, \quad (3.3.5.14)$$

obtained that ${}_q\mathcal{L}_s\{\varphi(t)/t\}$ exists.

Property 11:

$${}_q\mathcal{L}_s\left\{\int_t^{\infty} \frac{\varphi(\tau)}{\tau} d_q\tau\right\} = \frac{1}{s} \int_0^{qs} \Omega(\tau) d_q\tau, \quad (3.3.5.15)$$

Provided that

- i) the two integrals exist,
- ii) $\Omega(s)$ exists for every s ,
- iii) $q^j \sum_{r=1}^{\infty} |\varphi(q^j/s)| = O(|h|^j)$, for all fixed j and $|h| < 1$.

Property 12:

$${}_q\mathcal{L}_s\{I_q^m \varphi(t)\} = \left(\frac{1-q}{s}\right)^m \Omega(s), \quad (3.3.5.16)$$

where I_q^m is defined in (3.3.5.7).

Property 13: If ${}_q\mathcal{L}_s\{\varphi_k(t)\} = \Omega_k(s)$, then

$${}_q\mathcal{L}_s\left\{\sum_{k=0}^m \varphi_k(t)\right\} = \sum_{k=0}^m \Omega_k(s), \quad (3.3.5.17)$$

once at least one of the following conditions hold,

- a) m is finite
- b) m is infinite and
 - i) $\sum_{k=0}^{\infty} |\varphi_k(tq^j)|$ is convergent for every $t_0 q^j$, t_0 being fixed,
 - ii) $q^j \sum_{k=0}^{\infty} |\varphi_k(tq^j)| = O(h^j)$, where j is greater than some fixed J and h is a fixed quantity, $|h|$ being less than unity.

Property 14: If $\lim_{t \rightarrow 0} \frac{\varphi(t)}{t} = 0$ then,

$${}_q\mathcal{L}_s\left\{\int_0^t \frac{\varphi(t)}{t} d_q t\right\} = \frac{1}{s} \int_{qs}^{\infty} \Omega(\tau) d_q\tau. \quad (3.3.5.18)$$

Property 15: If ${}_q\mathcal{L}_s\{\varphi(t)\} = \Omega(s)$, then

$${}_q\mathcal{L}_s\{t^m \varphi(t)\} = (q-1)^m D_{qs}^m \Omega(s), \quad (3.3.5.19)$$

$${}_q\mathcal{L}_s\{t^m D_q^n \varphi(t)\} = (-q^{-1})^n (q-1)^{m-n} D_{qs}^m (s^n \Omega(s/q^n)) \text{ for } n \leq m, \quad (3.3.5.20)$$

and

$${}_q\mathcal{L}_s\{t^m D_q^n \varphi(t)\} = (q-1)^m D_q^n \left[\frac{s^n}{(q(1-q))^n} \Omega(s/q^n) - \sum_{k=1}^n \frac{s^{k-1} D_q^{n-k} \varphi(0)}{q^{k-1} (1-q)^k} \right], \quad (3.3.5.21)$$

for every $n > m$.

3.4 THE Q-FRACTIONAL CALCULUS

3.4.1 Classical Fractional Calculus

In this part, we present the continuous case of the fractional calculus version. $\varphi \in \mathcal{AC}^{(m)}[a, b]$ if φ has continuous derivatives up to order $m-1$ on $[a, b]$ with $\varphi^{(m-1)} \in \mathcal{AC}[a, b]$.

Lemma 3.4.1.1: Let $\mathcal{AC}^{(m)}[a, b]$ exist the functions φ giving in the form,

$$\varphi(t) = \sum_{k=0}^{m-1} c_k (t-a)^k + \frac{1}{(m-1)!} \int_a^t (t-\tau)^{m-1} \Omega(\tau) d\tau,$$

where $\Omega \in L_1(a, b)$ and the c_k 's are arbitrary constants (see [37]). In addition to this,

$$\Omega(t) = \varphi^{(m)}(t), \text{ and } c_t = \frac{\varphi^{(t)}(a)}{t!}, \quad t = 0, 1, \dots, m-1.$$

The first fractional Riemann-Liouville integral operator is related to Abel's integral equation,

$$\frac{1}{\Gamma(\gamma)} \int_a^t (t-\tau)^{\gamma-1} \Omega(\tau) d\tau = \varphi(t), \quad t > a, \quad \gamma > 0, \quad \varphi \in L_1(a, b) \quad (3.4.1.1)$$

Theorem 3.4.1.1 (3.4.1.1) with $0 < \gamma < 1$ has a unique solution in $L_1(a, b)$ if and only if $\varphi_{1-\gamma}$ is written as

$$\varphi_{1-\gamma}(t) = \frac{1}{\Gamma(1-\gamma)} \int_a^t (t-\tau)^{-\gamma} \varphi(\tau) d\tau$$

and $\varphi_{1-\gamma}$ is absolutely continuous on $[a, b]$ such that $\varphi_{1-\gamma}(a) = 0$. Therefore, Ω can be shown directly as,

$$\Omega(t) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_a^t (t-\tau)^{-\gamma} \varphi(\tau) d\tau = \frac{d}{dt} \varphi_{1-\gamma}(t), \quad (3.4.1.2)$$

when $\varphi \in \mathcal{AC}[a, b]$, then $\varphi_{1-\gamma} \in \mathcal{AC}[a, b]$ and also (3.4.1.2) becomes

$$\Omega(t) = \frac{1}{\Gamma(1-\gamma)} \left[\frac{\varphi(a)}{(t-a)^\gamma} + \int_a^t \frac{\varphi'(s)}{(t-s)^\gamma} ds \right].$$

Similarly, n -th primitive of a function $\varphi \in L_1(a, b)$ can be used with Cauchy formulae (see [37]),

$$\int_a^t \int_a^{t_{m-1}} \dots \int_a^{t_1} \varphi(\tau) d\tau dt_1 \dots dt_{m-1} = \frac{1}{(m-1)!} \int_a^t (t-\tau)^{m-1} \varphi(\tau) d\tau \quad (3.4.1.3)$$

$$\int_t^b \int_{t_{m-1}}^b \dots \int_{t_1}^b \varphi(\tau) d\tau dt_1 \dots dt_{m-1} = \frac{1}{(m-1)!} \int_t^b (\tau-t)^{m-1} \varphi(\tau) d\tau \quad (3.4.1.4)$$

and $m \in \mathbb{N}$. The right hand sides of (3.4.1.3) and (3.4.1.4) hold for non integer values of m , the Riemann-Liouville fractional integral operator can be given when

$\gamma \in \mathbb{R}^+$ and $\varphi \in L_1(a, b)$;

$$I_{a+}^\gamma \varphi(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-\tau)^{\gamma-1} \varphi(\tau) d\tau \quad (3.4.1.5)$$

$$I_{b-}^\gamma \varphi(t) = \frac{1}{\Gamma(\gamma)} \int_t^b (\tau-t)^{\gamma-1} \varphi(\tau) d\tau$$

and $t \in (a, b)$.

When $\varphi \in L_1(a, b)$, then not only $I_{a+}^\gamma \varphi$ but also $I_{b-}^\gamma \varphi$ exist and they are $L_1(a, b)$ functions. In addition, if $\varphi \in L_1(a, b)$, the below expression is obtained

$$\lim_{\gamma \rightarrow 0^+} I_{a+}^\gamma \varphi(t) = \lim_{\gamma \rightarrow 0^+} I_{b-}^\gamma \varphi(t) = \varphi(t), \quad (3.4.1.6)$$

For $\varphi \in L_1(a, b)$, the left and right sided Riemann-Liouville fractional derivatives of order γ , $\gamma \in \mathbb{R}^+$, are defined formally by

$$D_{a+}^\gamma \varphi(t) = D^p I_{a+}^{p-\gamma} \varphi(t) = \frac{1}{\Gamma(p-\gamma)} \frac{d^p}{dt^p} \int_a^t (t-\tau)^{p-\gamma-1} \varphi(\tau) d\tau. \quad (3.4.1.7)$$

Now, in this part, some properties of Riemann-Liouville fractional calculus are indicated which their q-analogues are derived. Especially, consider the case of the left-sided Riemann-Liouville fractional operator because, its basic analogue will be studied. Let $\gamma, \theta \in \mathbb{R}^+$ and if $\varphi \in L_1(a, b)$ then the semigroup property

$$I_{a+}^\gamma I_{a+}^\theta \varphi(t) = I_{a+}^\theta I_{a+}^\gamma \varphi(t) = I_{a+}^{\gamma+\theta} \varphi(t), \quad (3.4.1.8)$$

holds for almost every $t \in [a, b]$. When $\varphi(t)$ satisfies the conditions

$\varphi \in L_1(a, b)$ and $I_{a+}^{p-\gamma} \varphi \in \mathcal{AC}^{(p)}[a, b]$, $p = [\gamma]$ then,

$$\begin{cases} D_{a+}^{\gamma-j} \varphi \in L_1(a, b), & j = 0, 1, \dots, p, \\ D_{a+}^{\gamma-j} \varphi \in \mathcal{AC}^{(j)}[a, b], & j = 1, 2, \dots, p-1. \end{cases} \quad (3.4.1.9)$$

In addition,

$$D_{a+}^{\gamma} I_{a+}^{\gamma} \varphi(t) = \varphi(t) \quad (3.4.1.10)$$

$$D_{a+}^{\gamma} I_{a+}^{\theta} \varphi(t) = I_{a+}^{\theta-\gamma} \varphi(t) \quad , \theta \geq \gamma \geq 0, \quad (3.4.1.11)$$

$$D_{a+}^{\gamma} I_{a+}^{\theta} \varphi(t) = D_{a+}^{\gamma-\theta} \varphi(t) \quad , \gamma > \theta \geq 0, \quad (3.4.1.12)$$

Also,

$$I_{a+}^{\gamma} D_{a+}^{\gamma} \varphi(t) = \varphi(t) - \sum_{j=1}^p D_{a+}^{\gamma-j} \varphi(a^+) \frac{(t-a)^{\gamma-j}}{\Gamma(1+\gamma-j)}, \quad (3.4.1.13)$$

When $\varphi \in L_1(a, b)$ and $D_{a+}^{-(m-\theta)} \varphi \in \mathcal{AC}^{(m)}[a, b]$, $m = [\theta]$, then the expression

$$I_{a+}^{\gamma} D_{a+}^{\theta} \varphi(t) = D_{a+}^{\theta-\gamma} \varphi(t) - \sum_{j=1}^p D_{a+}^{\theta-j} \varphi(a^+) \frac{(t-a)^{\gamma-j}}{\Gamma(1+\gamma-j)}, \quad (3.4.1.14)$$

holds almost everywhere in (a, b) for any $\gamma > 0$. Also, when $0 \leq p-1 \leq \gamma < p$,

$\gamma + \theta < p$ and $D_{a+}^{-(m-\gamma)} \varphi \in \mathcal{AC}^{(m)}[a, b]$ then,

$$D_{a+}^{\gamma} D_{a+}^{\theta} \varphi(t) = D_{a+}^{\gamma+\theta} \varphi(t) - \sum_{j=1}^m D_{a+}^{\theta-j} \varphi(a^+) \frac{(t-a)^{-\gamma-j}}{\Gamma(1-\gamma-j)} \quad (3.4.1.15)$$

holds almost everywhere in (a, b) . Finally,

$I_{a+}^{\gamma} D_{a+}^{\gamma} \varphi(t) = \varphi(t)$ and $I_{a+}^{\gamma} D_{a+}^{\theta} \varphi(t) = D_{a+}^{\theta-\gamma} \varphi(t)$, where $\varphi \in I_{a+}^{\gamma}(L_1)$ and $\varphi \in I_{a+}^{\theta}(L_1)$. Many fractional operators were defined in [1]. Especially, the Caputo-fractional operator is introduced as,

$${}^c_a D_t^{\gamma} \varphi(t) = \frac{1}{\Gamma(m-\gamma)} \int_a^t (t-\tau)^{m-\gamma-1} \varphi^{(m)}(\tau) d\tau$$

and $m-1 < \gamma \leq m$, $m \in \mathbb{N}$. From [37], the Riemann-Liouville integral and the Caputo fractional derivative can be related by,

$${}^c_a D_t^{\gamma} \varphi(t) = I_{a+}^{m-\gamma} \varphi^{(m)}(t).$$

If the function $\varphi(t)$ has $m+1$ continuous derivatives in $[a, b]$, then [63],

$$\lim_{\gamma \rightarrow m} {}^c_a D_t^{\gamma} \varphi(t) = \varphi^{(m)}(t) \text{ for } t \in [a, b].$$

While constant c is put instead of the function φ when the Caputo-fractional derivative is taken then,

$$\begin{aligned}
D_{a+}^{\gamma} c &= D^m I_{a+}^{m-\gamma} c \\
&= \frac{c}{\Gamma(m-\gamma+1)} D^m (t-a)^{m-\gamma} \\
&= \frac{c}{\Gamma(1-\gamma)} (t-a)^{-\gamma},
\end{aligned}$$

where $m = \lceil \gamma \rceil$.

3.4.2 q-fractional Riemann-Liouville Operator

The q-fractional Riemann-Liouville integral is introduced [37] as

$$I_q^{\theta} \varphi(t) = \frac{t^{\theta-1}}{\Gamma_q(\theta)} \int_0^t (q\tau/t; q)_{\theta-1} \varphi(\tau) d_q \tau$$

And the q-fractional Riemann-Liouville derivative is given below

$$D_q^{\theta} \varphi(t) = D_q^{[\theta]} I_q^{\theta-[\theta]} \varphi(t), \quad (\theta \geq 0). \quad (3.4.2.1)$$

Properties of the q-fractional operators

Lemma 3.4.2.1 [37] If $\varphi \in L_q^1[0, b]$ then (3.4.2.1) holds

Proof: Initially, the proof is done by [37]

$$I_q^{\gamma} \varphi(t) = t^{\gamma} (1-q)^{\gamma} \sum_{m=0}^{\infty} q^m \frac{(q^{\gamma}; q)_m}{(q; q)_m} \varphi(tq^m). \quad (3.4.2.2)$$

Therefore,

$$I_q^{\gamma} (I_q^{\theta} \varphi(t)) = t^{\gamma+\theta} (1-q)^{\gamma+\theta} \sum_{k=0}^{\infty} q^{k(1+\theta)} \frac{(q^{\gamma}; q)_k}{(q; q)_k} \sum_{n=0}^{\infty} q^n \frac{(q^{\theta}; q)_n}{(q; q)_n} \varphi(tq^{m+n}).$$

If the substitution $m = k + n$ is used, then the following expression is obtained.

$$\begin{aligned}
&I_q^{\gamma} (I_q^{\theta} \varphi(t)) \\
&= t^{\gamma+\theta} (1-q)^{\gamma+\theta} \sum_{k=0}^{\infty} q^{k(1+\theta)} \frac{(q^{\gamma}; q)_k}{(q; q)_k} \sum_{m=k}^{\infty} q^{m-k} \frac{(q^{\theta}; q)_{m-k}}{(q; q)_{m-k}} \varphi(tq^m). \quad (3.4.2.3)
\end{aligned}$$

Because $\varphi \in L_q^1[0, b]$, the sums in (3.4.2.3) is absolutely convergent. Thus, the order of summations can be interchanged to satisfy, namely

$$I_q^{\gamma} (I_q^{\theta} \varphi(t)) = t^{\gamma+\theta} (1-q)^{\gamma+\theta} \sum_{m=0}^{\infty} q^m \varphi(tq^m) \sum_{k=0}^m q^{k\theta} \frac{(q^{\gamma}; q)_k}{(q; q)_k} \frac{(q^{\theta}; q)_{m-k}}{(q; q)_{m-k}}.$$

It is easy to see that, [37];

$$\frac{(q^{\theta}; q)_{m-k}}{(q; q)_{m-k}} = \frac{(q^{-m}; q)_k}{(q^{1-m-\theta}; q)_k} q^{(1-\theta)k}$$

So it is concluded that,

$$\begin{aligned} & I_q^\gamma (I_q^\theta \varphi(t)) \\ &= t^{\gamma+\theta} (1-q)^{\gamma+\theta} \sum_{m=0}^{\infty} q^m \varphi(tq^m) \sum_{k=0}^{\infty} q^{k\theta} \frac{(q^\gamma; q)_k}{(q; q)_k} \frac{(q^{-m}; q)_k}{(q^{1-m-\theta}; q)_k} q^{(1-\theta)k}. \end{aligned}$$

As a result,

$$\begin{aligned} & I_q^\gamma (I_q^\theta \varphi(t)) \\ &= t^{\gamma+\theta} (1-q)^{\gamma+\theta} \sum_{m=0}^{\infty} q^m \varphi(tq^m) \frac{(q^\theta; q)_m}{(q; q)_m} {}_2\phi_1 (q^{-m}, q^\gamma; q^{1-m-\theta}; q, q). \end{aligned} \quad (3.4.2.4)$$

Hence,

$${}_2\phi_1 (q^{-m}, q^\gamma; q^{1-m-\theta}; q, q) = \frac{(q^{1-m-\theta-\gamma}; q)_m}{(q^{1-m-\theta}; q)_m} q^{m\gamma} = \frac{(q^{\gamma+\theta}; q)_m}{(q^\theta; q)_m}. \quad (3.4.2.5)$$

Lemma 3.4.2.2: When $\varphi \in \mathcal{L}_q^1[0, b]$ then

$$D_q^\gamma I_q^\gamma \varphi(t) = \varphi(t), \quad (\gamma > 0; t \in (0, b]). \quad (3.4.2.6)$$

Proof: If $\gamma = m$, $m \in \mathbb{N}$, then $D_q^m I_q^m \varphi(t) = \varphi(t)$. [37] If γ is a nonpositive integer such that, $m-1 < \gamma < m$, $m \in \mathbb{N}$, the semigroup property can be applied to satisfy $D_q^\gamma I_q^\gamma \varphi(t) = D_q^m I_q^{m-\gamma} I_q^\gamma \varphi(t) = D_q^m I_q^m \varphi(t) = \varphi(t)$, for every $t \in (0, b]$. ■

Lemma 3.4.2.3: If $\gamma \in \mathbb{R}^+$ and $m := [\gamma]$. Also when $\varphi \in L_q^1[0, b]$, such that $I_q^{m-\gamma} \varphi \in \mathcal{AC}_q^{(m)}[0, b]$, then,

$$I_q^\gamma D_q^\gamma \varphi(t) = \varphi(t) - \sum_{p=1}^m D_q^{\gamma-p} \varphi(0^+) \frac{t^{\gamma-p}}{\Gamma_q(\gamma-p+1)}, \quad t \in (0, b]. \quad (3.4.2.7)$$

Proof: We start with,

$$h(\tau, t) = t^\gamma (q\tau/t; q)_\gamma D_q^\gamma \varphi(\tau), \quad t \in (0, b], \quad (0 < \tau \leq t).$$

So, it is said that $h(t, gt) = 0$ for all $t \in (0, b]$ (see [37]). Besides,

$$\begin{aligned} I_q^\gamma D_q^\gamma \varphi(t) &= \frac{t^{\gamma-1}}{\Gamma_q(\gamma)} \int_0^t (q\tau/t; q)_{\gamma-1} D_q^\gamma \varphi(\tau) d_q \tau \\ &= D_{q,t} \left(\frac{t^\gamma}{\Gamma_q(\gamma+1)} \int_0^t (q\tau/t; q)_\gamma D_q^\gamma \varphi(\tau) d_q \tau \right) \\ &= D_{q,t} \left(\frac{t^\gamma}{\Gamma_q(\gamma+1)} \int_0^t (q\tau/t; q)_\gamma D_q^m I_q^{m-\gamma} \varphi(\tau) d_q \tau \right) \end{aligned} \quad (3.4.2.8)$$

and the q -integration is used n times on the last q -integral of (3.4.2.8), then,

$$\int_0^t \psi(\tau) D_q \varphi(\tau) d_q \tau = (\varphi \psi)(t) - \lim_{m \rightarrow \infty} (\varphi \psi)(t q^m) - \int_0^t D_q \psi(\tau) \varphi(q\tau) d_q \tau$$

Later,

$$\frac{t^\gamma}{\Gamma_q(\gamma + 1)} \int_0^t (q\tau/t; q)_\gamma D_q^\gamma \varphi(\tau) d_q \tau$$

is shown in details in the next page. This calculation are introduced as,

$$\begin{aligned} & \frac{t^\gamma}{\Gamma_q(\gamma + 1)} \int_0^t (q\tau/t; q)_\gamma D_q^\gamma \varphi(\tau) d_q \tau \\ &= - \sum_{p=1}^m D_{q,t}^{\gamma-p} \varphi(0^+) \frac{t^{\gamma-p+1}}{\Gamma_q(\gamma - p + 2)} \\ & \quad + \frac{t^{\gamma-m}}{\Gamma_q(\gamma - m + 1)} \int_0^t (q\tau/t; q)_{\gamma-m} I_q^{m-\gamma} \varphi(\tau) d_q \tau \\ &= - \sum_{p=1}^m D_{q,t}^{\gamma-p} \varphi(0^+) \frac{t^{\gamma-p+1}}{\Gamma_q(\gamma - p + 2)} + I_q^{\gamma-m+1} I_q^{m-\gamma} \varphi(t). \end{aligned} \quad (3.4.2.9)$$

After (3.4.2.1) and (3.4.2.2) are used, then it is satisfied that

$$\begin{aligned} & \frac{t^\gamma}{\Gamma_q(\gamma + 1)} \int_0^t (q\tau/t; q)_\gamma D_q^\gamma \varphi(\tau) d_q \tau \\ &= I_q \varphi(t) - \sum_{p=1}^m D_q^{\gamma-p} \varphi(0^+) \frac{t^{\gamma-p+1}}{\Gamma_q(\gamma - p + 2)}. \end{aligned} \quad (3.4.2.10)$$

According to (3.4.2.7), $I_q^\gamma D_q^\gamma \varphi(t) = \varphi(t)$, ($t \in (0, b]$) holds if and only if

$$D_q^{\gamma-p} \varphi(0^+) = 0, \quad (p = 1, 2, \dots, m).$$

Lemma 3.4.2.4: If $\varphi \in L_q^1[0, b]$, then

$$D_q^\gamma I_q^\theta \varphi(t) = I_q^{\theta-\gamma} \varphi(t), \quad (\theta \geq \gamma \geq 0; t \in (0, b]), \quad (3.4.2.11)$$

In addition to this, if $D_q^{\gamma-\theta} \varphi(t)$ exists in $(0, b]$ then,

$$D_q^\gamma I_q^\theta \varphi(t) = D_q^{\gamma-\theta} \varphi(t), \quad (\gamma > \theta \geq 0). \quad (3.4.2.12)$$

Proof: First of all, we consider that $\theta \geq \gamma$. Later, $\theta = \gamma + (\theta - \gamma)$ and from (3.3.1) the proof is introduced by (see [37])

$$D_q^\gamma I_q^\theta \varphi(t) = D_q^\gamma I_q^\gamma I_q^{\theta-\gamma} \varphi(t) = I_q^{\theta-\gamma} \varphi(t),$$

and if $\theta < \gamma$, $n = [\gamma]$ and $m = [\gamma - \theta]$

The following is obtained.

$$\begin{aligned} D_q^\gamma I_q^\theta \varphi(t) &= D_q^n I_q^{n-\gamma} I_q^\theta \varphi(t) = D_q^n I_q^{n-m} I_q^{m-\gamma+\theta} \varphi(t) \\ &= D_q^m I_q^{m-\gamma+\theta} \varphi(t) = D_q^{\gamma-\theta} \varphi(t). \end{aligned} \quad \blacksquare$$

Lemma 3.4.2.5: When $\varphi \in \mathcal{L}_q^1[0, b]$, such that $I_q^{m-\theta} \varphi \in \mathcal{AC}_q^{(m)}[0, b]$, where $\theta > 0$ and $m = [\theta]$, (see [37]). Later, for every, $\gamma \geq 0$ then

$$I_q^\gamma D_q^\theta \varphi(t) = D_q^{-\gamma+\theta} \varphi(t) - \sum_{p=1}^m D_q^{\theta-p} \varphi(0^+) \frac{t^{\gamma-p}}{\Gamma_q(\gamma-p+1)}, \quad t \in (0, b]. \quad (3.4.2.13)$$

Proof: It is known that

$$\frac{1}{\Gamma_q(z)}$$

has zeros at the negative integers and (3.4.2.13) exists for every $\theta > 0$, when $\gamma = 0$, [37]. So, we consider that $\gamma > 0$. Especially, we assume below two different cases.

First Case: If $\gamma \geq \theta$, then this condition is satisfied,

$$\begin{aligned} I_q^\gamma D_q^\theta \varphi(t) &= I_q^{\gamma-\theta} (I_q^\theta D_q^\theta \varphi(t)) \\ &= I_q^{\gamma-\theta} \left(\varphi(t) - \sum_{p=1}^m D_q^{\theta-p} \varphi(0^+) \frac{t^{\theta-p}}{\Gamma_q(\theta-p+1)} \right) \\ &= I_q^{\gamma-\theta} \varphi(t) - \sum_{p=1}^m D_q^{\theta-p} \varphi(0^+) \frac{t^{\gamma-p}}{\Gamma_q(\gamma-p+1)} \end{aligned}$$

for all $t \in (0, b]$, (see [37]).

Second case: If $\theta \geq \gamma$, then this condition is obtained,

$$\begin{aligned} I_q^\gamma D_q^\theta \varphi(t) &= D_q^{\theta-\gamma} (I_q^\theta D_q^\theta \varphi(t)) \\ &= D_q^{\theta-\gamma} \left(\varphi(t) - \sum_{p=1}^m D_q^{\theta-p} \varphi(0^+) \frac{t^{\theta-p}}{\Gamma_q(\theta-p+1)} \right), \\ &= D_q^{\theta-\gamma} \varphi(t) - \sum_{p=1}^m D_q^{\theta-p} \varphi(0^+) \frac{t^{\gamma-p}}{\Gamma_q(\gamma-p+1)} \end{aligned}$$

for every $t \in (0, b]$ (see [37]).

3.4.3 q-fractional Caputo Operator

Definition 3.4.3.1: For $\gamma > 0$, the q-fractional Caputo operator is given by (see [37]);

$${}^c D_q^\gamma \varphi(t) = I_q^{m-\gamma} D_q^m \varphi(t), \quad m := [\gamma], \quad (3.4.3.1)$$

when γ is nonnegative integer then the operator is introduced by

$${}^c D_q^m \varphi(t) = I_q^0 D_q^m \varphi(t) = D_q^m \varphi(t).$$

Theorem 3.4.3.2: Let $\gamma > 0$ and $m = [\gamma]$. When $\varphi \in \mathcal{AC}_q^{(m)}[0, b]$, then ${}^c D_q^\gamma \varphi(t) \in \mathcal{L}_q^1[0, b]$.

Proof: when $\varphi \in \mathcal{AC}_q^{(m)}[0, b]$, then it is clear that $D_q^{(m)} \varphi \in \mathcal{L}_q^1[0, b]$. Thus, we conclude

$${}^c D_q^\gamma \varphi(t) = I_q^{m-\gamma} D_q^m \varphi(t) \in \mathcal{L}_q^1[0, b]. \quad \blacksquare$$

Theorem 3.4.3.3: Let γ and θ be positive numbers and if $m = [\gamma]$ and $n = [\theta]$ (see [37]), then

1. When $\varphi \in \mathcal{AC}_q^{(n)}[0, b]$, then [37];

$$I_q^\gamma {}^c D_q^\theta \varphi(t) = \begin{cases} I_q^{\theta-\gamma} \varphi(t) - \sum_{p=0}^{n-1} \frac{D_q^p \varphi(0^+)}{\Gamma_q(\theta - \gamma + p + 1)} t^{\theta-\gamma+p}, & \theta \geq \gamma, \\ D_q^{\theta-\gamma} \varphi(t) - \sum_{p=0}^{m-1} \frac{D_q^p \varphi(0^+)}{\Gamma_q(\theta - \gamma + p + 1)} t^{\theta-\gamma+p}, & \theta < \gamma, \end{cases} \quad (3.4.3.2)$$

for every $t \in (0, b]$.

2. When $\varphi \in \mathcal{L}_q^1[0, b]$ such that $I_q^\gamma \varphi \in \mathcal{AC}_q^{(n)}[0, b]$, then [37];

$${}^c D_q^\theta I_q^\gamma \varphi(t) = \begin{cases} I_q^{\gamma-\theta} \varphi(t), & \gamma \geq n \geq \theta, \\ D_q^{\theta-\gamma} \varphi(t) - \sum_{p=0}^{[n-\gamma]} \frac{D_q^{n-\gamma-p} \varphi(0^+)}{\Gamma_q(n - \theta - p + 1)} t^{n-\theta-p}, & \theta > \gamma, \end{cases} \quad (3.4.3.3)$$

for every $t \in (0, b]$.

Proof: Initially we show (3.4.2) by (see [37]);

$$I_q^\gamma {}^c D_q^\theta \varphi(t) = I_q^\gamma I_q^{n-\theta} D_q^n \varphi(t) = I_q^{n+\gamma-\theta} D_q^n \varphi(t).$$

Then (3.4.2) follows for $\gamma \geq n \geq \theta$ by using,

$D_q^\gamma I_q^\theta \varphi(t) = I_q^{\theta-\gamma} \varphi(t)$ and $\varphi \in \mathcal{L}_q^1[0, a]$ and $\theta > \gamma$. Later, if we use

$$D_q^\gamma I_q^\theta \varphi(t) = D_q^{\gamma-\theta} \varphi(t), \quad (\gamma > \theta \geq 0),$$

where γ and θ are interchanged by $n + \gamma - \theta$ and n . Taking into account that

$${}^c D_q^\theta I_q^\gamma \varphi(t) = I_q^{n-\theta} D_q^n I_q^\gamma \varphi(t),$$

We reach at,

$$D_q^n I_q^\gamma \varphi(t) = \begin{cases} I_q^{\gamma-n}, & \gamma \geq n \geq \theta, \\ D_q^{n-\gamma}, & \gamma < n. \end{cases}$$

Therefore, when $\gamma \geq n \geq \theta$ from the semigroup property (see [37]);

$${}^c D_q^\theta I_q^\gamma \varphi(t) = I_q^{n-\theta} I_q^{\gamma-n} \varphi(t) = I_q^{\gamma-\theta} \varphi(t).$$

if $\gamma < n$, then

$${}^c D_q^\theta I_q^\gamma \varphi(t) = I_q^{n-\theta} D_q^{n-\gamma} \varphi(t).$$

■

3.4.4 q-Laplace transform of fractional q-Integrals and q-Derivatives

In this part, q-Laplace transform are shown for the Riemann-Liouville fractional q-integral and q-derivatives.

Theorem 3.4.4.1: In [37], if $\bar{\varphi} \in \ell_q^1[0, b]$ and $\Omega(s) = {}_q L_s\{\bar{\varphi}(t)\}$ then,

$${}_q L_s\{I_q^\theta \bar{\varphi}(t)\} = \frac{(1-q)^\theta}{s^\theta} \Omega(s), \quad \forall \theta > 0, \quad (3.4.4.1)$$

when $m-1 < \theta \leq m$ and $I_q^{m-\theta} \bar{\varphi}(t) \in \mathcal{AC}_q^{(m)}[0, b]$ then,

$${}_q L_s\{D_q^\theta \bar{\varphi}(t)\} = \frac{s^\theta}{(1-q)^\theta} \Omega(s) - \sum_{k=1}^m D_q^{\theta-k} \bar{\varphi}(0^+) \frac{s^{k-1}}{(1-q)^k}, \quad (3.4.4.2)$$

Proof:

$$\begin{aligned} I_q^\theta \bar{\varphi}(t) &= (1-q) \frac{t^{\theta-1}}{\Gamma_q(\theta)} * {}_q \bar{\varphi}(t) \\ &= (1-q)(\Omega_{\theta-1}(t) * {}_q \bar{\varphi}(t)), \end{aligned}$$

(3.4.4.1) is satisfied from,

$$\Omega_\theta(t) = \frac{t^\theta}{\Gamma_q(\theta+1)}, \quad (\theta > -1)$$

then,

$${}_q L_s\{\Omega_\theta(t)\} = \frac{(1-q)^\theta}{s^{\theta+1}}, \quad \text{Re}(s) > 0,$$

$$(\bar{\varphi} * \bar{\psi}) = \frac{1}{1-q} \int_0^t \bar{\varphi}(\tau) \varepsilon^{-q\tau} \bar{\psi}(\tau) d_q \tau$$

and

$${}_q L_s\{\bar{\varphi} * \bar{\psi}\} = {}_q L_s \bar{\varphi} {}_q L_s \bar{\psi}.$$

Now, (3.4.4.2) was proven from,

$$D_q^\theta \varphi(t) = \Omega(t) = D_q^k I_q^{k-\theta} \varphi(t), \quad (k = [\theta]),$$

$${}_q L_s \{D_q^\theta \varphi(t)\} = \left(\frac{s}{1-q}\right)^\theta \Omega(s) - \sum_{k=1}^m D_q^{m-k} \varphi(0) \frac{s^{k-1}}{(1-q)^k}, \quad (m \in \mathbb{N}),$$

and

$${}_q L_s \{I_q^\theta \bar{\varphi}(t)\} = \frac{(1-q)^\theta}{s^\theta} \Omega(s), \quad (\theta > 0).$$

We obtain,

$$\begin{aligned} {}_q L_s \{D_q^\theta \bar{\varphi}(t)\} &= {}_q L_s \{D_q^k I_q^{k-\theta} \bar{\varphi}(t)\} \\ &= \frac{s^k}{(1-q)^k} {}_q L_s \{I_q^{k-\theta} \bar{\varphi}(t)\} - \sum_{k=1}^m D_q^{m-k} I_q^{m-\theta} \bar{\varphi}(0^+) \frac{s^{k-1}}{(1-q)^k} \\ &= \frac{s^\theta}{(1-q)^\theta} \Omega(s) - \sum_{k=1}^m D_q^{\theta-k} \bar{\varphi}(0^+) \frac{s^{k-1}}{(1-q)^k}, \end{aligned} \quad (3.4.4.3)$$

■

Theorem 3.4.4.2: [37] If ${}_q L_s \{\varphi(t)\} = \Omega(s)$ then the q-Laplace transform of the Caputo fractional q-derivative is introduced by,

$${}_q L_s \{ {}^c D_q^\theta \varphi(t) \} = \frac{s^\theta}{(1-q)^\theta} \left(\Omega(s) - \sum_{k=0}^{m-1} D_q^k \varphi(0^+) \frac{(1-q)^k}{s^{k+1}} \right)$$

Proof: Because, ${}^c D_q^\theta \varphi(t) = (1-q) D_q^m \varphi(t) * \Omega_{m-\theta-1}(t)$ then by,

$$\Omega_\theta(t) = \frac{t^\theta}{\Gamma_q(\theta+1)}, \quad (\theta > -1)$$

then,

$${}_q L_s \{\Omega_\theta(t)\} = \frac{(1-q)^\theta}{s^{\theta+1}}, \quad \operatorname{Re}(s) > 0$$

$${}_q L_s \{D_q^\theta \varphi(t)\} = \left(\frac{s}{1-q}\right)^\theta \Omega(s) - \sum_{k=1}^m D_q^{m-k} \varphi(0) \frac{s^{k-1}}{(1-q)^k}, \quad (m \in \mathbb{N})$$

then

$$\begin{aligned} {}_q L_s \{ {}^c D_q^\theta \varphi(t) \} &= \frac{(1-q)^{m-\theta}}{s^{m-\theta}} {}_q L_s \{D_q^m \varphi(t)\} \\ &= \frac{(1-q)^{m-\theta}}{s^{m-\theta}} \left(\left(\frac{s}{1-q}\right)^m \Omega(s) - \sum_{k=1}^m D_q^{m-k} \varphi(0^+) \frac{s^{k-1}}{(1-q)^k} \right) \end{aligned}$$

$$= \left(\frac{s^\theta}{(1-q)^\theta} \Omega(s) - \sum_{k=0}^{m-1} D_q^k \varphi(0^+) \frac{s^{-k+\theta-1}}{(1-q)^{-k+\theta}} \right)$$

■

Lemma 3.4.4.1: [37] If ${}_q L_S \{\varphi(t)\} = \Omega(s)$ then q-Laplace transform of the Riemann-Liouville sequential q-derivative of order $k\theta$, $0 < \theta < 1$, is defined by,

$${}_q L_S \{\mathfrak{Q}_q^{k\theta} \varphi(t)\} = r^{k\theta} \Omega(s) - \frac{1}{1-q} \sum_{k=0}^{m-1} r^{\theta(m-1-k)} I_q^{1-\theta} D_q^{\theta p} \varphi(0^+).$$

and $r = \frac{s}{1-q}$. When $\mathfrak{Q}_q^{k\theta}$ is given in;

$$\mathfrak{Q}_q^\theta y = D_q^\theta y, \quad \mathfrak{Q}_q^{k\theta} y = D_q^\theta \mathfrak{Q}_q^{(k-1)\theta} y, \quad (k \in \mathbb{N}).$$

Lemma 3.4.4.2: [37] If ${}_q L_S \{\varphi(t)\} = \Omega(s)$, then the q-Laplace transform of the Caputo sequential q-derivative of order $k\theta$, $0 < \theta < 1$, is introduced as,

$${}_q L_S \{ {}^C \mathfrak{Q}_q^{k\theta} \varphi(t) \} = r^{k\theta} \Omega(s) - \frac{1}{1-q} \sum_{k=0}^{m-1} r^{\theta(m-1-k)} {}^C \mathfrak{Q}_q^{\theta p} \varphi(0^+).$$

Lemma 3.4.4.3: In [37], let $\theta, \gamma, a \in \mathbb{R}^+$ and $n \in \mathbb{N}$. So the expression is

$${}_q L_S \{ t^{n\theta+\gamma-1} e_{\theta,\gamma}^{(n)}(\pm at^\theta; q) \} = \frac{r^{\theta-\gamma}}{1-q} \frac{n!}{(r^\theta \mp a)^{n+1}}, \quad |r|^\theta > a, \quad (3.4.4.4)$$

is valid in the disc $\{t \in \mathbb{C}: a|t(1-q)|^\theta < 1\}$.

Proof: From (3.3.4.3) then we satisfy,

$$e_{\theta,\gamma}^{(n)}(v; q) = \sum_{k=0}^{\infty} \frac{k(k-1) \dots (k-n+1)}{\Gamma_q(\theta k + \gamma)} v^{k-n}, \quad |v| < (1-q)^{-\theta},$$

So, for $|at^\theta| < (1-q)^{-\theta}$ we satisfy,

$$t^{n\theta+\gamma-1} e_{\theta,\gamma}^{(n)}(\pm at^\theta; q) = \sum_{k=0}^{\infty} \frac{k(k-1) \dots (k-n+1)}{\Gamma_q(\theta k + \gamma)} (\pm a)^{k-n} t^{\theta k + \gamma - 1},$$

Hence,

$$\begin{aligned} {}_q L_S \{ t^{n\theta+\gamma-1} e_{\theta,\gamma}^{(n)}(\pm at^\theta; q) \} &= \sum_{k=0}^{\infty} k(k-1) \dots (k-n+1) (\pm a)^{k-n} \frac{(1-q)^{\theta k + \gamma - 1}}{s^{\theta k + \gamma}} \\ &= \frac{r^{-n\theta-\gamma}}{1-q} \sum_{k=0}^{\infty} k(k-1) \dots (k-n+1) (\pm ar^{-\theta})^{k-n} \\ &= \frac{r^{-n\theta-\gamma}}{1-q} \frac{d^n}{d\xi^n} \sum_{k=0}^{\infty} \xi^k, \quad \xi = \pm ar^{-\theta}. \end{aligned}$$

By using $a^{-1}|r|^\theta > 1$ then we satisfy, $|\xi| = |ar^{-\theta}| < 1$. Because,

$$\begin{aligned} \frac{r^{-n\theta-\gamma}}{1-q} \frac{d^n}{d\xi^n} \sum_{k=0}^{\infty} \xi^k &= \frac{r^{-n\theta-\gamma}}{1-q} \frac{d^n}{d\xi^n} \frac{1}{1-\xi} \\ &= \frac{r^{-n\theta-\gamma}}{1-q} \frac{n!}{(1-\xi)^{n+1}} \\ &= \frac{r^{-n\theta-\gamma}}{1-q} \frac{n!}{(1 \mp ar^{-\theta})^{n+1}} \\ &= \frac{r^{\theta-\gamma}}{1-q} \frac{n!}{(r^\theta \mp a)^{n+1}}. \end{aligned} \quad \blacksquare$$

Lemma 3.4.4.4: Now, consider $\Omega_{\theta,l}(t, \mu_i)$ is the function introduced for $l = 1, 2, \dots, \sigma_i$, $i = 1, \dots, k$ and $\mu_i \in \mathbb{C}$ by,

$$\Omega_{\theta,l}(t, \mu_i) = t^{l\theta+\theta-1} \sum_{k=0}^{\infty} (k+l)(k+l-1) \dots (k+1) \frac{(\mu_i t^\theta)^k}{\Gamma_q(k\theta + l\theta + \theta)},$$

$i = 1, \dots, K$, $l = 1, \dots, \sigma_i$ and $|\mu_i||t(1-q)|^\theta < 1$. Then,

$${}_qL_s\{\Omega_{\theta,l}(t, \mu_i)\} = \frac{l!}{1-q} (r^\theta - \mu_i)^{-l-1},$$

holds for $|r|^\theta > |\mu_i|$ when $r = \frac{s}{1-q}$, [42].

Proof: By using the properties of the q-Laplace transform (see [37]), the following is obtained

$$\begin{aligned} {}_qL_s\{\Omega_{\theta,l}(t, \mu_i)\} &= {}_qL_s\left\{\sum_{k=0}^{\infty} (k+l)(k+l-1) \dots (k+1) \frac{(\mu_i)^k t^{k\theta+l\theta+\theta-1}}{\Gamma_q(k\theta + l\theta + \theta)}\right\} \\ &= \frac{r^{-l\theta-\theta}}{1-q} \sum_{k=0}^{\infty} (k+l)(k+l-1) \dots (k+1) (\mu_i r^{-\theta})^k. \end{aligned}$$

So, for $|\mu_i| < r^\theta$, then

$$\begin{aligned} {}_qL_s\{\Omega_{\theta,l}(t, \mu_i)\} &= \frac{r^{-l\theta-\theta}}{1-q} \frac{d^l}{dz^l} \sum_{k=0}^{\infty} z^k \Big|_{z=\mu_i r^{-\theta}}, \\ &= \frac{r^{-l\theta-\theta}}{1-q} \frac{d^l}{dz^l} \frac{1}{1-z} \Big|_{z=\mu_i r^{-\theta}}, \\ &= \frac{l!}{(1-q)(r^\theta - \mu_i)^{l+1}}. \end{aligned} \quad \blacksquare$$

CHAPTER IV

Q-PROPORTIONAL DERIVATIVE AND APPLICATION OF Q-LAPLACE TRANSFORM

In this chapter, we focus on the definition of the q-proportional derivative.

4.1 DEFINITION OF THE Q-PROPORTIONAL DERIVATIVE

In this part, q-proportional derivative will be introduced. The discrete version of proportional derivative was applied with coronavirus model by [64]. Also, it was applied to other fields of engineering models in [65]. Similarly, the q-proportional derivative can be related with control systems, dynamical systems and mathematical biology. Also, q-Laplace transforms were showed to solve both q-proportional derivative and q-fractional proportional derivative in this thesis.

Initially, q-proportional derivative is defined as;

$$\mathbf{D}_q\varphi(t) = \kappa_1(\beta, t)\varphi(t) + \kappa_0(\beta, t)D_q\varphi(t) \quad (4.1.1)$$

and $\beta \in [0,1]$. With this information, it is concluded that $\kappa_0, \kappa_1: [0,1] \times \mathbb{R} \rightarrow [0, \infty)$ be continuous such that for every $t \in \mathbb{R}$, [21, 22].

$$\lim_{\beta \rightarrow 0^+} \kappa_1(\beta, t) = 1,$$

$$\lim_{\beta \rightarrow 0^+} \kappa_0(\beta, t) = 0,$$

$$\lim_{\beta \rightarrow 1^-} \kappa_1(\beta, t) = 0,$$

$$\lim_{\beta \rightarrow 1^-} \kappa_0(\beta, t) = 1,$$

and $\kappa_1(\beta, t) \neq 0, \beta \in [0,1), \kappa_0(\beta, t) \neq 0, \beta \in (0,1]$. Specifically, the case is assumed when $\kappa_1(\beta, t) = 1 - \beta$ and $\kappa_0(\beta, t) = \beta$. Then, (4.1.1) becomes

$$\mathbf{D}_q\varphi(t) = (1 - \beta)\varphi(t) + \beta D_q\varphi(t). \quad (4.1.2)$$

Notably, $\lim_{\beta \rightarrow 0^+} \mathbf{D}_q\varphi(t) = \varphi(t)$ and $\lim_{\beta \rightarrow 1^-} \mathbf{D}_q\varphi(t) = D_q\varphi(t)$.

Also, ($0 < q < 1$ and $0 < \beta < 1$ in (4.1.2). Now, $\mathbf{D}_q^2\varphi(t), \mathbf{D}_q^3\varphi(t), \dots, \mathbf{D}_q^m\varphi(t)$ can be modified in (4.1.2) but m is higher order of q -derivative. The second, third and higher order terms are given as;

$$\mathbf{D}_q(\mathbf{D}_q\varphi(t)) = (1 - \beta)D_q\varphi(t) + \beta D_q^2\varphi(t)$$

then

$$\mathbf{D}_q^2\varphi(t) = (1 - \beta)[(1 - \beta)\varphi(t) + \beta D_q\varphi(t)] + \beta[(1 - \beta)D_q\varphi(t) + \beta D_q^2\varphi(t)],$$

so we say,

$$\mathbf{D}_q^2\varphi(t) = (1 - \beta)^2\varphi(t) + \beta(1 - \beta)D_q\varphi(t) + \beta(1 - \beta)D_q\varphi(t) + \beta^2 D_q^2\varphi(t),$$

$$\mathbf{D}_q^2\varphi(t) = (1 - \beta)^2\varphi(t) + 2\beta(1 - \beta)D_q\varphi(t) + \beta^2 D_q^2\varphi(t). \quad (4.1.3)$$

Specifically, the second order q -derivative in (4.1.3) is identified with summation formula. (4.1.3) and (4.1.4) are similar. Therefore, (4.1.3) can be written as;

$$\mathbf{D}_q^2\varphi(t) = \sum_{k=0}^2 \binom{2}{k} (1 - \beta)^{2-k} \beta^k D_q^{(k)}\varphi(t) \quad (4.1.4)$$

and ($0 < q, \beta < 1$).

The form of $\mathbf{D}_q^3\varphi(t)$ is given as;

$$\mathbf{D}_q^3\varphi(t) = (1 - \beta)^2 D_q\varphi(t) + 2\beta(1 - \beta)D_q^2\varphi(t) + \beta^2 D_q^3\varphi(t), \text{ or}$$

$$\mathbf{D}_q^3\varphi(t) = (1 - \beta)^2 D_q\varphi(t) + 2\beta(1 - \beta)D_q(D_q\varphi(t)) + \beta^2 D_q(D_q^2\varphi(t))$$

then

$$\begin{aligned} \mathbf{D}_q^3\varphi(t) &= (1 - \beta)^2[(1 - \beta)\varphi(t) + \beta D_q\varphi(t)] + \\ &+ 2\beta(1 - \beta)[(1 - \beta)D_q\varphi(t) + \beta D_q^2\varphi(t)] + \beta^2[(1 - \beta)D_q^2\varphi(t) + \beta D_q^3\varphi(t)] \end{aligned}$$

and later,

$$\begin{aligned} \mathbf{D}_q^3\varphi(t) &= (1 - \beta)^3\varphi(t) + \beta(1 - \beta)^2 D_q\varphi(t) + 2\beta(1 - \beta)^2 D_q\varphi(t) + \\ &+ 2\beta^2(1 - \beta)D_q^2\varphi(t) + \beta^2(1 - \beta)D_q^2\varphi(t) + \beta^3 D_q^3\varphi(t). \end{aligned}$$

Finally, the new form of $\mathbf{D}_q^3\varphi(t)$ is found in (4.1.5).

$$\begin{aligned} \mathbf{D}_q^3\varphi(t) &= (1 - \beta)^3\varphi(t) + 3\beta(1 - \beta)^2 D_q\varphi(t) \\ &+ 3\beta^2(1 - \beta)D_q^2\varphi(t) + \beta^3 D_q^3\varphi(t). \end{aligned} \quad (4.1.5)$$

Also, as mentioned above, (4.1.5) could be denoted as;

$$\mathbf{D}_q^3\varphi(t) = \sum_{k=0}^3 \binom{3}{k} (1 - \beta)^{3-k} \beta^k D_q^k\varphi(t). \quad (4.1.6)$$

Similarly, fourth-order q -derivative can be defined with summation formula as below;

$$\mathbf{D}_q^4 \varphi(t) = \sum_{k=0}^4 \binom{4}{k} (1-\beta)^{4-k} \beta^k D_q^k \varphi(t) \quad (4.1.7)$$

and higher order q-derivative is written as below;

$$\mathbf{D}_q^m \varphi(t) = \sum_{k=0}^m \binom{m}{k} (1-\beta)^{m-k} \beta^k D_q^k \varphi(t). \quad (4.1.8)$$

4.2 SOLUTION OF Q-PROPORTIONAL DIFFERENTIAL EQUATIONS WITH Q-LAPLACE TRANSFORM

This section is about solution of q-proportional derivative and q-Laplace transform. Previously, q-Laplace transformation is introduced by [66]. Generally, by using the Laplace transformation, we can solve easily many differential equations. Similarly, by using the q-Laplace transformation we can solve q-proportional differential equations. Besides, q-Laplace transform is shown for the considered function and higher order derivatives of the function as;

$${}_q \mathcal{L}_s \{\varphi(bt)\} = (1/b) \Omega(s/b), \quad (b \neq 0)$$

so

$${}_q \mathcal{L}_s \{\varphi(t)\} = \Omega(s) \quad (4.2.1)$$

and

$$\begin{aligned} {}_q \mathcal{L}_s \{D_q^m \varphi(t)\} &= \left(\frac{s}{1-q}\right)^m \Omega(s) \\ &\quad - \sum_{k=1}^m D_q^{m-k} \varphi(0) \frac{s^{k-1}}{(1-q)^k}, \quad (m \in \mathbb{N}). \end{aligned} \quad (4.2.2)$$

By using (4.2.2) for $m = 1$ we satisfy;

$${}_q \mathcal{L}_s \{D_q \varphi(t)\} = \left(\frac{s}{1-q}\right) \Omega(s) - \frac{\varphi(0)}{1-q}, \quad (q \neq 1). \quad (4.2.3)$$

Similarly, we get q-Laplace transformation of q-derivative and it says ($0 < q < 1$) in (4.2.3).

Now, we define the q-proportional derivative in order to apply q-Laplace transform and solve it easily. By using (4.1.2), we investigate the following q-differential equation

$$\mathbf{D}_q \varphi(t) := (1-\beta)\varphi(t) + \beta D_q \varphi(t) = \psi(t). \quad (4.2.4)$$

Proof of (4.2.4): When q-Laplace transformation is used with (4.2.4) then, the proof is shown as,

$${}_q\mathcal{L}_s\{\psi(t)\} = {}_q\mathcal{L}_s\{(1 - \beta)\varphi(t) + \beta D_q\varphi(t)\},$$

or

$${}_q\mathcal{L}_s\{\psi(t)\} = (1 - \beta) {}_q\mathcal{L}_s\{\varphi(t)\} + \beta {}_q\mathcal{L}_s\{D_q\varphi(t)\}.$$

$\Psi(s)$ is obtained as,

$$\Psi(s) = (1 - \beta)\Omega(s) + \beta \left[\frac{s}{1 - q} \Omega(s) - \frac{\varphi(0)}{1 - q} \right].$$

After some standard calculations this new expressions are done

$$\Psi(s) = (1 - \beta)\Omega(s) + \frac{\beta s}{1 - q} \Omega(s) - \frac{\beta\varphi(0)}{1 - q}.$$

As a result, it is reported that

$$\Psi(s) = \left[\frac{(1 - \beta)(1 - q) + \beta s}{1 - q} \right] \Omega(s) - \frac{\beta\varphi(0)}{1 - q},$$

$$\Psi(s) + \frac{\beta\varphi(0)}{1 - q} = \left[\frac{(1 - \beta)(1 - q) + \beta s}{1 - q} \right] \Omega(s), \text{ respectively.}$$

Finally, the following is obtained, namely

$$\frac{(1 - q)\Psi(s) + \beta\varphi(0)}{1 - q} = \left[\frac{(1 - \beta)(1 - q) + \beta s}{1 - q} \right] \Omega(s),$$

for $(0 < q < 1)$.

Then, it is concluded

$$\begin{aligned} \Omega(s) &= \frac{(1 - q)\Psi(s) + \beta\varphi(0)}{(1 - \beta)(1 - q) + \beta s} \\ &= \frac{(1 - q)\Psi(s)}{(1 - \beta)(1 - q) + \beta s} + \frac{\beta\varphi(0)}{(1 - \beta)(1 - q) + \beta s} \\ &= \frac{(1 - q)\Psi(s)}{\beta \left(s + \frac{(1 - \beta)(1 - q)}{\beta} \right)} + \frac{\beta\varphi(0)}{\beta \left(s + \frac{(1 - \beta)(1 - q)}{\beta} \right)}. \end{aligned}$$

Now the inverse q-Laplace transform is applied to find the expression of $\varphi(t)$, namely

${}_q\mathcal{L}_s^{-1}\{\Omega(s)\} = \varphi(t)$. After some calculations it is the followings is reported

$$\begin{aligned} {}_q\mathcal{L}_s^{-1}\{\Omega(s)\} &= {}_q\mathcal{L}_s^{-1} \left\{ \frac{(1 - q)\Psi(s)}{\beta \left(s + \frac{(1 - \beta)(1 - q)}{\beta} \right)} \right\} \\ &+ {}_q\mathcal{L}_s^{-1} \left\{ \frac{\beta\varphi(0)}{\beta \left(s + \frac{(1 - \beta)(1 - q)}{\beta} \right)} \right\}. \end{aligned}$$

Later, $\alpha = \frac{(1-\beta)(1-q)}{\beta}$, ($\beta > 1$), ($0 < q < 1$) and ($\alpha < 0$) are assumed to show step by step easily. So, φ is reported as,

$$\varphi(t) = \left(\frac{1-q}{\beta}\right) {}_q\mathcal{L}_s^{-1}\left\{\frac{\Psi(s)}{s-\alpha}\right\} + {}_q\mathcal{L}_s^{-1}\left\{\frac{\varphi(0)}{s-\alpha}\right\}$$

and

$$\varphi(t) = \frac{(1-q)}{\beta} (\psi(t) * e_q(\alpha t)) + \varphi(0)e_q(\alpha t), \text{ respectively.}$$

We recall that the convolution was introduced and inverse q-Laplace transformation were mentioned in [37]-[67]. So, the following is obtained

$$\varphi(t) = \frac{1-q}{\beta} \int_0^t \psi(\tau - \alpha) d_q \tau + \varphi(0)e_q(\alpha t). \quad (4.2.5)$$

CHAPTER V

Q-FRACTIONAL PROPORTIONAL DERIVATIVE

The original contributions of this thesis are reported below.

5.1 DEFINITION OF Q-FRACTIONAL PROPORTIONAL DERIVATIVE

In this section, we mention q-fractional proportional derivative where the Caputo q-fractional operator will be used, namely

$${}^{CP}\mathbf{D}_q^\theta \varphi(t) = (1 - \beta)\varphi(t) + \beta {}^C D_q^\theta \varphi(t), \quad (0 < \theta < 1).$$

Similarly, the q-Laplace transform was identified for Caputo version in [66] and q-Laplace transform of q-fractional Caputo derivative is given as

$${}_q\mathcal{L}_s\{{}^C D_q^\theta \varphi(t)\} = \frac{s^\theta}{(1-q)^\theta} {}_q\mathcal{L}_s\{\varphi(t)\} - \frac{\varphi(0)s^{\theta-1}}{(1-q)^\theta}, \quad (0 < \theta < 1)$$

and

$${}_q\mathcal{L}_s\{\varphi(t)\} = \Omega(s).$$

Then, from the above expression, (5.1.1) was obtained as,

$${}_q\mathcal{L}_s\{{}^C D_q^\theta \varphi(t)\} = \frac{s^\theta}{(1-q)^\theta} \Omega(s) - \frac{\varphi(0)s^{\theta-1}}{(1-q)^\theta}, \quad (0 < \theta < 1), \quad (5.1.1)$$

In the next part, we investigate the solution of the following differential equation, as an example

$${}^{CP}\mathbf{D}_q^\theta \varphi(t) = (1 - \beta)\varphi(t) + \beta {}^C D_q^\theta \varphi(t) = \psi(t), \quad (0 < \theta < 1). \quad (5.1.2)$$

The detailed steps of finding the solution of (5.1.2) is given below:

We start with

$${}_q\mathcal{L}_s\{\psi(t)\} = {}_q\mathcal{L}_s\{(1 - \beta)\varphi(t) + \beta {}^C D_q^\theta \varphi(t)\},$$

Thus, we conclude that

$$\Psi(s) = (1 - \beta) {}_q\mathcal{L}_s\{\varphi(t)\} + \beta {}_q\mathcal{L}_s\{{}^C D_q^\theta \varphi(t)\}.$$

After some sample derivations we report that

$$\Psi(s) = (1 - \beta)\Omega(s) + \beta \left[\frac{s^\theta}{(1 - q)^\theta} \Omega(s) - \frac{\varphi(0)s^{\theta-1}}{(1 - q)^\theta} \right]$$

or

$$\Psi(s) = \Omega(s) \left[(1 - \beta) + \frac{\beta s^\theta}{(1 - q)^\theta} \right] - \frac{\beta \varphi(0)s^{\theta-1}}{(1 - q)^\theta}.$$

As a result, we conclude that

$$\Psi(s) + \frac{\beta \varphi(0)s^{\theta-1}}{(1 - q)^\theta} = \Omega(s) \left[(1 - \beta) + \frac{\beta s^\theta}{(1 - q)^\theta} \right].$$

Thus, we report

$$\frac{\Psi(s)(1 - q)^\theta + \beta \varphi(0)s^{\theta-1}}{(1 - q)^\theta} = \Omega(s) \left[\frac{(1 - \beta)(1 - q)^\theta + \beta s^\theta}{(1 - q)^\theta} \right].$$

From the above expression we get

$$\Omega(s) = \frac{\Psi(s)(1 - q)^\theta + \beta \varphi(0)s^{\theta-1}}{(1 - q)^\theta} \frac{(1 - q)^\theta}{(1 - \beta)(1 - q)^\theta + \beta s^\theta}$$

or

$$\Omega(s) = \frac{\Psi(s)(1 - q)^\theta + \beta \varphi(0)s^{\theta-1}}{(1 - \beta)(1 - q)^\theta + \beta s^\theta},$$

respectively. Rearranging the terms we conclude that

$$\Omega(s) = \frac{\Psi(s)(1 - q)^\theta}{(1 - \beta)(1 - q)^\theta + \beta s^\theta} + \frac{\beta \varphi(0)s^{\theta-1}}{(1 - \beta)(1 - q)^\theta + \beta s^\theta}$$

or

$$\Omega(s) = \frac{\Psi(s)(1 - q)^\theta}{\beta \left(s^\theta + \frac{(1 - \beta)(1 - q)^\theta}{\beta} \right)} + \frac{\beta \varphi(0)s^{\theta-1}}{\beta \left(s^\theta + \frac{(1 - \beta)(1 - q)^\theta}{\beta} \right)}.$$

Now, we consider that $\alpha = \frac{(1 - \beta)(1 - q)^\theta}{\beta}$ and $(0 < \theta < 1, 0 < q < 1)$, then $\alpha > 0$. So,

we conclude

$$\Omega(s) = \frac{\Psi(s)(1 - q)^\theta}{\beta(s^\theta + \alpha)} + \frac{\varphi(0)s^{\theta-1}}{(s^\theta + \alpha)}. \quad (5.1.3)$$

The inverse q-Laplace transform was used for (5.1.3), namely

$$\begin{aligned} {}_q\mathcal{L}_s^{-1}\{\Omega(s)\} &= {}_q\mathcal{L}_s^{-1}\left\{ \frac{\Psi(s)(1 - q)^\theta}{\beta(s^\theta + \alpha)} + \frac{\varphi(0)s^{\theta-1}}{(s^\theta + \alpha)} \right\} \\ &= \frac{(1 - q)^\theta}{\beta} {}_q\mathcal{L}_s^{-1}\left\{ \frac{\Psi(s)}{s^\theta + \alpha} \right\} + \varphi(0) {}_q\mathcal{L}_s^{-1}\left\{ \frac{s^{\theta-1}}{s^\theta + \alpha} \right\}. \end{aligned}$$

Convolution is used in here in order to introduce the inverse q-Laplace transform. Particularly, convolution was reported in [37]. By using the q-Laplace transformation, we obtained q-convolution

$$\varphi(t) = \frac{(1-q)^\theta}{\beta} \left[{}_q\mathcal{L}_s^{-1} \left\{ {}_q\mathcal{L}_s\{\psi(t)\} * {}_q\mathcal{L}_s\{(1-q)t^{\theta-1}e_{\theta,\theta}^{(0)}(-\alpha t^\theta; q)\} \right\} \right] + \varphi(0)e_{\theta,1}^{(0)}(-\alpha t^\theta; q)(1-q).$$

Therefore we conclude

$$\varphi(t) = \frac{(1-q)^\theta}{\beta} \left[{}_q\mathcal{L}_s^{-1} \left({}_q\mathcal{L}_s\{\psi(t) * (1-q)t^{\theta-1}e_{\theta,\theta}^{(0)}(-\alpha t^\theta; q)\} \right) \right] + \varphi(0)e_{\theta,1}^{(0)}(-\alpha t^\theta; q)(1-q).$$

The q-convolution formula was obtained from above steps of solutions

$$\begin{aligned} \varphi(t) &= \frac{(1-q)^\theta}{\beta} \left(\psi(t) * (1-q)t^{\theta-1}e_{\theta,\theta}^{(0)}(-\alpha t^\theta; q) \right) \\ &\quad + \varphi(0)e_{\theta,1}^{(0)}(-\alpha t^\theta; q)(1-q). \end{aligned} \quad (5.1.4)$$

Finally, q-convolution formula of (5.1.4) is written as;

$$\begin{aligned} \varphi(t) &= \left(\frac{(1-q)^\theta}{\beta} \right) \left(\frac{1}{1-q} \right) \int_0^t \psi(\tau) \varepsilon^{-q\tau} (1-q)\tau^{\theta-1} e_{\theta,\theta}^{(0)}(-\alpha\tau^\theta; q) d_q\tau \\ &\quad + \varphi(0)e_{\theta,1}^{(0)}(-\alpha t^\theta; q)(1-q), \end{aligned}$$

or

$$\begin{aligned} \varphi(t) &= \left(\frac{(1-q)^\theta}{\beta} \right) \int_0^t \psi(\tau) \varepsilon^{-q\tau} \tau^{\theta-1} e_{\theta,\theta}^{(0)}(-\alpha\tau^\theta; q) d_q\tau \\ &\quad + \varphi(0)e_{\theta,1}^{(0)}(-\alpha t^\theta; q)(1-q). \end{aligned} \quad (5.1.5)$$

Hence, the solution of a q-fractional proportional differential equation in Caputo sense given by equation (5.1.2) is derived, as an illustrative example.

CHAPTER VI

CONCLUSION

In this thesis, the definitions and some properties of q-derivative and q-integral were reviewed. After that, the q-analogue of the proportional derivative is mentioned. Next, the fractional counterpart of the q-proportional derivatives are discussed. Thereafter, for the first time, the q-analogue of the proportional derivative was introduced. More precisely, we defined the following operator:

$${}^{CP}\mathbf{D}_q^\theta \varphi(t) = (1 - \beta)\varphi(t) + \beta {}^C D_q^\theta \varphi(t), \quad (0 < \theta < 1).$$

The solutions of some q-differential equations in the frame of q-proportional fractional derivatives were found with the assistance of the q-Laplace transforms.

It is worth mentioning that since the operators discussed in this thesis, any qualitative results such as Gronwall inequality and Lyapunov inequality may be obtained for related q-difference operators. On the top of this, optimal control and biological problems can be reformulated in the framework of such operators.

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APPENDICES

APPENDIX 1: Our used formulas of q-Laplace transforms and inverse q-Laplace transforms are given below:

1. $\mathcal{L}_q\{\varphi(t)\} = \Omega(s)$
2. $\mathcal{L}_q\{D_q\varphi(t)\} = \left(\frac{s}{1-q}\right)\Omega(s) - \frac{\varphi(0)}{1-q}, \quad (q \neq 1, 0 < q < 1)$
3. $\mathcal{L}_q\{\varphi(bt)\} = (1/b)\Omega(s/b), \quad (b \neq 0)$
4. $\mathcal{L}_q\{D_q^m\varphi(t)\} = \left(\frac{s}{1-q}\right)^m \Omega(s) - \sum_{k=1}^m D_q^{m-k}\varphi(0) \frac{s^{k-1}}{(1-q)^k}, \quad (m \in \mathbb{N})$
5. $\mathcal{L}_q^{-1}\left\{\frac{\varphi(0)}{s-\alpha}\right\} = \varphi(0)e_q(\alpha t)$
6. $\mathcal{L}_q^{-1}\left\{\frac{\Psi(s)}{s-\alpha}\right\} = \psi(t) * e_q(\alpha t) = \int_0^t \psi(t-\alpha)d_q\tau$
7. $\mathcal{L}_q\{ {}^c D_q^\theta \varphi(t)\} = \frac{s^\theta}{(1-q)^\theta} \Omega(s) - \frac{\varphi(0)s^{\theta-1}}{(1-q)^\theta}, \quad (0 < \theta < 1, \quad 0 < q < 1)$
8. $\mathcal{L}_q^{-1}\left\{\frac{s^{\theta-1}}{s^\theta+\alpha}\right\} = \varphi(0)e_{\theta,1}^{(0)}(-\alpha t^\theta; q)(1-q), \quad (0 < \theta < 1)$
9. $\mathcal{L}_q^{-1}\left\{\frac{\Psi(s)}{s^\theta+\alpha}\right\} = \psi(t) * (1-q)t^{\theta-1}e_{\theta,\theta}^{(0)}(-\alpha t^\theta; q)$

$$= \frac{1}{1-q} \int_0^t \psi(\tau) \varepsilon^{-q\tau} (1-q)\tau^{\theta-1} e_{\theta,\theta}^{(0)}(-\alpha\tau^\theta; q) d_q\tau$$

$$= \int_0^t \psi(\tau) \varepsilon^{-q\tau} \tau^{\theta-1} e_{\theta,\theta}^{(0)}(-\alpha\tau^\theta; q) d_q\tau.$$